INMAS 2021, Modeling & Optimization Problems: Session 3.

1. In the lecture, we learned that simplex starts at a vertex of the feasible region and travels to adjacent vertices. However, what do we do if we don't know any feasible solutions? It is possible to find a feasible solution by solving a different LP. Assume that we are given an LP in standard form:

$$\max c^{\intercal} x$$

s.t.

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Here, the vector b may contain some negative components, in which case it is not there is no obvious feasible solution. Formulate a new LP such that:

- The new LP always has a feasible solution, and you can give a simple, closed form for that solution.
- The new LP takes an objective value of zero if the original LP is feasible.
- The new LP has an objective value of greater than zero if the original LP is infeasible.
- Given a solution to the new LP with objective value of zero, you can construct a feasible solution to the original LP.

This LP can be solved to identify a feasible solution.

Solution. One such LP is as follows:

$$\max \sum_{i=1}^{n} -y_i$$

s.t.

$$\begin{aligned} Ax - y &\leq b \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Here, the vector x represents a solution to the original LP. The variable y_i represents the amount by which the solution x exceeds constraint i. We will show that this LP satisfies all four desired properties:

- There is always a feasible solution. One example is $x = 0$ and y is defined so that

$$y_i = \begin{cases} -b & \text{if } b \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that the original LP has a feasible solution $\tilde{x}$. We must show that the new LP has optimal value of zero. The solution given by $x = \tilde{x}$, $y = 0$ is feasible for the new LP. The objective of this solution is zero, and no solution can have a an objective of greater than zero because the vector y is restricted to take non-negative values.
- Suppose that the original LP is infeasible, and let us show that the new LP will have an optimal value of less than zero. As discussed in the lecture, for any LP there are three exhaustive and mutually exclusive options: the LP can have an unbounded objective value, the LP can be infeasible, or the LP has an optimal. For the new LP, the objective value cannot be greater than zero, so it is not unbounded, and there is a feasible solution. Therefore, the LP must have an optimal solution (x^*, y^*). If y^* were identically zero, then x^* would be a feasible solution for the original LP. By assumption in this case, the original LP is feasible. Thus, y^* must have some strictly positive component. Then, the objective value must be less than zero.
- Suppose there is a solution (x^*, y^*) with objective value zero. Then, it must be true that y^* is identically zero. Then we can see that x^* is a feasible solution for the original LP.

□

2. The pigeonhole principle states that the problem, "Place $n + 1$ pigeons into n holes so that no two pigeons share a hole," has no solution.

   (a) Formulate this problem as an IP using the following two types of constraints:

      i. Those that enforce that every pigeon must be given a hole.

      ii. Those that enforce that, for each pair of pigeons, at most one of these pigeons can be assigned to a given hole.

   Show that the LP relaxation of this formulation is feasible.

   Solution. Decision variables:

$$x_{ij} - \text{binary; 1 if pigeon } i \text{ is assigned to hole } j$$

   Constraints:

   - Every pigeon must be given a hole:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \text{ in } 1, \ldots, n+1$$

- For each pair of pigeons, at most one of these pigeons can be assigned to a given hole:

$$x_{ij} + x_{kj} \leq 1 \text{ for all pairs } \{i, k\} \text{ in } 1, \ldots n+1; \text{ for } j \text{ in } 1, \ldots, n$$

An example feasible solution: set $x_{ij} = \dfrac{1}{n}$ for all i and all j.

□

(b) Alternatively, formulate this problem as an IP using the following two types of constraints:

i. Those that enforce that every pigeon must be given a hole.

ii. Those that enforce that every hole is assigned to at most one pigeon.

Show that the LP relaxation of this formulation is infeasible.

Solution. Decision variables:

$$x_{ij} - \text{binary; 1 if pigeon } i \text{ is assigned to hole } j$$

Constraints:

- Every pigeon must be given a hole:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \text{ in } 1, \ldots, n+1$$

- Every hole is assigned to at most one pigeon:

$$\sum_{i=1}^{n+1} x_{ij} \leq 1 \text{ for } j \text{ in } 1, \ldots, n$$

To show that this is infeasible, by summing all constraints of the first constraint, we see that it must be true:

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n} x_{ij} = n + 1$$

However, summing across the second constraints:

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n} x_{ij} \leq n$$

This is a clear contradiction, so there can be no feasible solution. □

3. In the lecture, we saw two formulations for the same facility location problem. In fact, one is stronger than the other. Formulation A has the following constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$
$$x_{ij} \leq y_j \text{ for all } i, j$$
$$x, y \geq 0$$
$$x_{ij}, y_j \in \mathbb{Z} \text{ for all } i, j$$

while Formulation B has the following constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$
$$\sum_{i=1}^{n} x_{ij} \leq n y_j \text{ for all } j$$
$$x, y \geq 0$$
$$x_{ij}, y_j \in \mathbb{Z} \text{ for all } i, j$$

Identify which formulation is stronger than the other, and prove it. (Assume that there are at least two possible facility locations; otherwise the problem is trivial and both LP relaxations are the same.)

Solution. Formulation A is stronger than formulation B. This is true because if

$$x_{ij} \leq y_j \text{ for all } i, j$$

Then, summing these constraints for $i = 1, \ldots, n$ gives us:

$$\sum_{i=1}^{n} x_{ij} \leq n y_j \text{ for all } i, j$$

Thus, any solution that satisfies the LP relaxation of Formulation A will also satisfy the LP relaxation of Formulation B.

Next, define a solution as follows, Let $x_{i1} = 1$ for all $i = 1, \ldots (n-1)$; $x_{n2} = 1$; $x_{ij} = 0$ for all other i, j. Let $y_1 = \frac{(n-1)}{n}$ and $y_2 = \frac{1}{n}$. Let $y_j = 0$ for all other j. Note that this solution is feasible for the LP relaxation of Formulation B, but not for Formulation A. □

4. Consider the following continuous knapsack problem:

$$\max_{x} \sum_{i=1}^{n} c_i x_i$$

s.t.

$$\sum_{i=1}^{n} a_i x_i \leq b$$
$$0 \leq x \leq 1$$

where a_i and c_i are positive numbers for each i, and b is a positive constant. Prove that the following greedy algorithm provides an optimal solution to this LP:

(a) Set $x_i = 0$ for all i. Set $r = b$; here b will represent the remaining weight. Let $I = \{1, \ldots, n\}$; here I will represent the set of remaining items.

(b) Let $t = \max_{i \in I} c_i/a_i$.

(c) Set $x_t = \min\{1, r/a_t\}$.

(d) Remove t from I.

(e) Set $r = r - a_t x_t$.

(f) If $r = 0$, then return x. Otherwise, go back to step (b).

Intuitively, this algorithm takes as much as possible of the item with the highest value-to-weight ratio until we have reached the maximum weight. Hint: form a feasible dual solution that achieves the same objective.

Solution. The dual is given by:

$$\min_{\lambda, \mu} b\lambda + \sum_{i=1}^{n} \mu_i$$

subject to:

$$a_i \lambda + \mu \geq c_i$$
$$\lambda, \mu \geq 0$$

where λ is a single variable and μ is a vector of n variables. Assume without loss of generality that $c_1/a_1 \geq c_2/a_2 \geq \ldots \geq c_n/a_n$ (this can always be achieved by relabeling variables). First, consider the case that $a_1 \geq b$. In this case, the solution produced by the greedy algorithm is $x_1 = b/a_1$ with all other variables equal to zero. An optimal dual solution is given by $\lambda = c_1/a_1$ and $\mu = 0$. It is straightforward to verify that the objectives of the primal solution and dual solution are equal, and that the primal solution is feasible. The dual solution is feasible because

$$a_i \lambda + \mu = a_i(c_1/a_1) = c_i \left(\frac{c_1/a_1}{c_i/a_i} \right) \geq c_i$$

Now, consider the case that $a_1 < b$. Let $\tau = \max\{i \in [n] \mid \sum_{i=1}^{\tau} a_i < b\}$. By assumption, since $a_1 < b$, the value τ exists. Verify that the greedy algorithm produces the following solution:

$$x_i = \begin{cases} 1 & \text{if } i \leq \tau \\ \dfrac{b - \sum_{i=1}^{\tau} a_i}{a_{\tau+1}} & \text{if } i = \tau + 1 \\ 0 & \text{if } i > \tau + 1 \end{cases}$$

Verify that this solution is feasible for the primal problem. Define a dual solution as follows:

$$\lambda = \frac{c_{t+1}}{a_{t+1}}$$
$$\mu_i = \begin{cases} c_i - a_i \dfrac{c_{t+1}}{a_{t+1}} & \text{if } i \leq t \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to verify that the dual solution achieves the same objective as the primal solution. It is clear that $\lambda \geq 0$; we must prove that $\mu \geq 0$. Let there be some $i \leq t$; then be assumption, $c_i/a_i \geq c_{t+1}/a_{t+1}$. This implies that

$$c_i - a_i \frac{c_{t+1}}{a_{t+1}} \geq 0$$

for any $i \leq t$. This in turn implies that $\mu \geq 0$. Now, we must show that

$$a_i \lambda + \mu \geq c_i$$

for each i. If $i \leq t$, then

$$a_i \lambda + \mu = a_i \frac{c_{t+1}}{a_{t+1}} + c_i - a_i \frac{c_{t+1}}{a_{t+1}} \qquad = c_i$$

If $i \geq t + 1$, then

$$a_i \lambda + \mu = a_i (c_{t+1}/a_{t+1}) = c_i \left(\frac{c_{t+1}/a_{t+1}}{c_i/a_i} \right) \geq c_i$$

This completes the proof.

□

5. Solve the following binary knapsack problem using branch-and-bound:

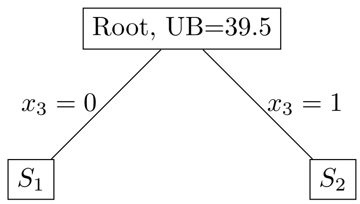
$$\max_x 17x_1 + 10x_2 + 25x_3 + 17x_4$$

s.t.

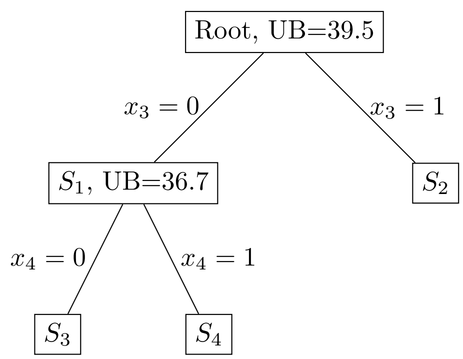
$$\sum_{i=1}^{4} 5x_1 + 3x_2 + 8x_3 + 7x_4 \leq 12$$
$$x_i \in \{0, 1\}$$

Solution. We start with a tree consisting of only the root node:

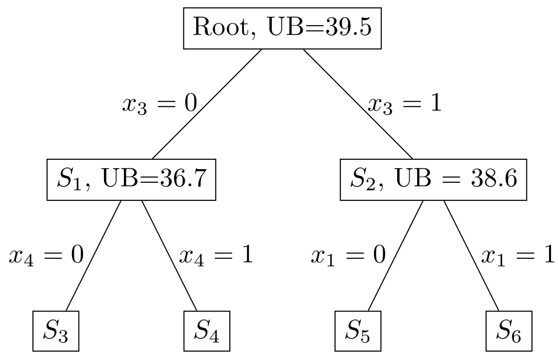
The solution at the root node is given by $(1, 1, 0.5, 0)$ with objective 39.5. Since x_3 is the only variable that takes a fractional value in the solution, it makes sense to branch on x_3. We add the branches $x_3 = 0$ and $x_3 = 1$, giving us the tree:



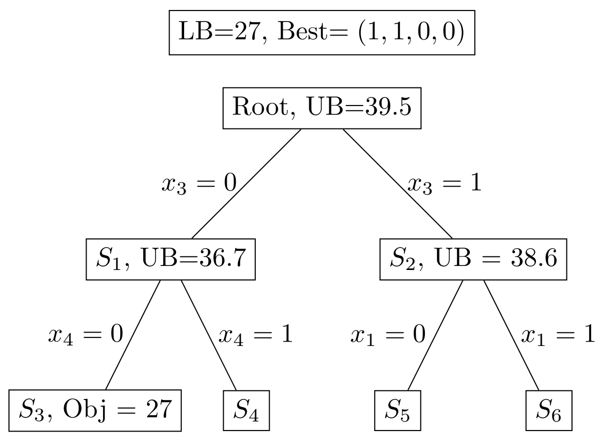
We could choose either the node S_1 or S_2 to evaluate; let's evaluate S_1. The solution at S_1 is given by $(1, 1, 0, 4/7)$ which has objective of 257/7 or approximately 36.7. It makes sense to branch on x_4, giving us the tree:



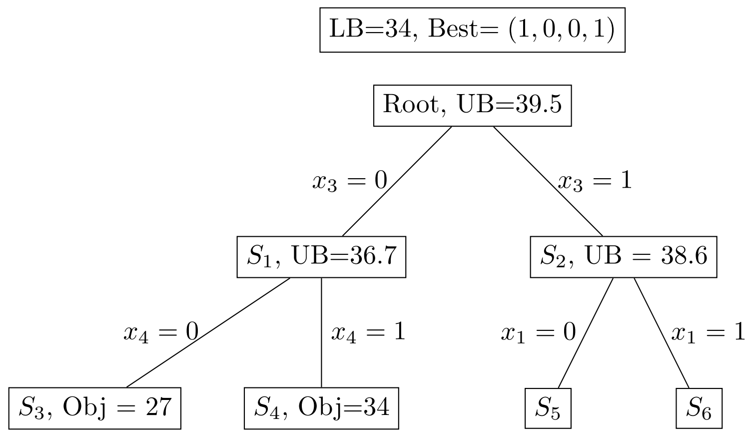
Now, we can choose the node S_2, S_3, or S_4 to evaluate. Let's choose S_2. The solution at S_2 is given by $(4/5, 0, 1, 0)$. This has an objective value of 142/5 or 38.6. It makes sense to branch on x_1, which gives us the tree:



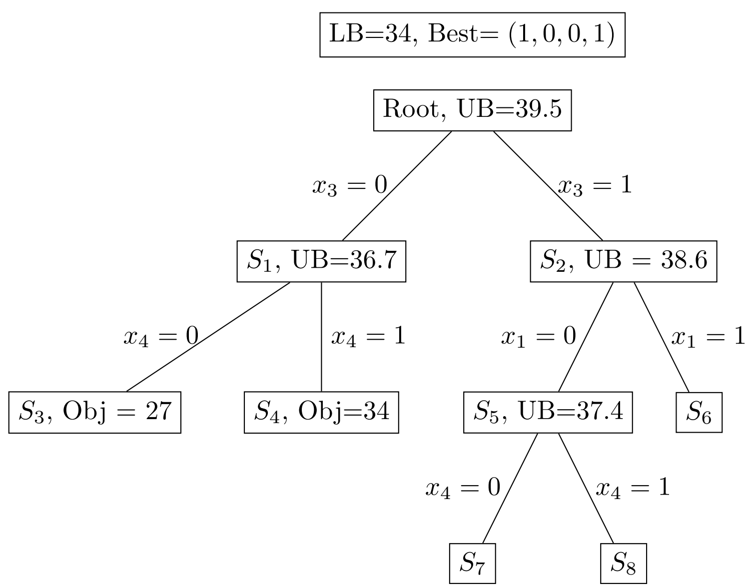
We can now evaluate any of the nodes S_3, S_4, S_5, S_6. Let's evaluate S_3. The optimal solution here is $(1, 1, 0, 0)$, which is feasible and achieves objective value of 27. This gives us a lower bound. There is no need to add more branches. The tree is now:



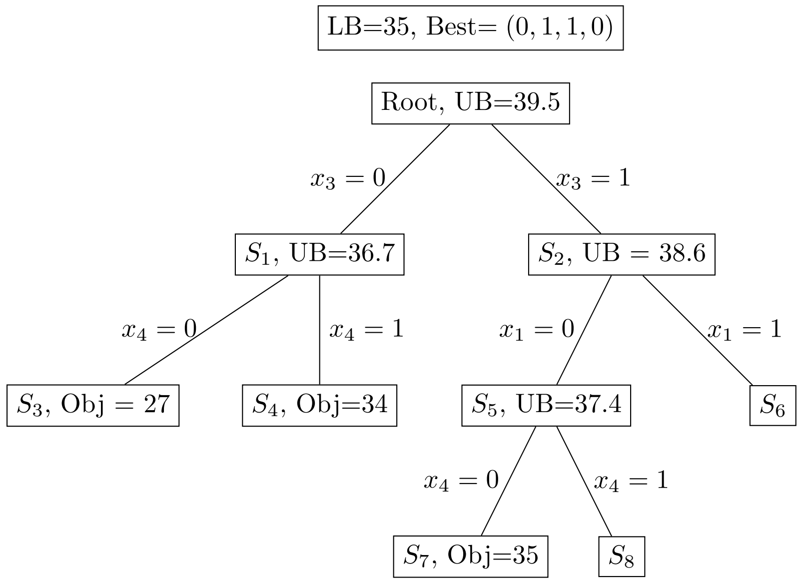
We can now evaluate S_4, S_5 or S_6. Let's choose S_4. The optimal solution at S_4 is $(1, 0, 0, 1)$. This is a feasible solution with objective value of 34. We update the lower bound and the best solution, and there is no branching needed at this node. The tree is now:



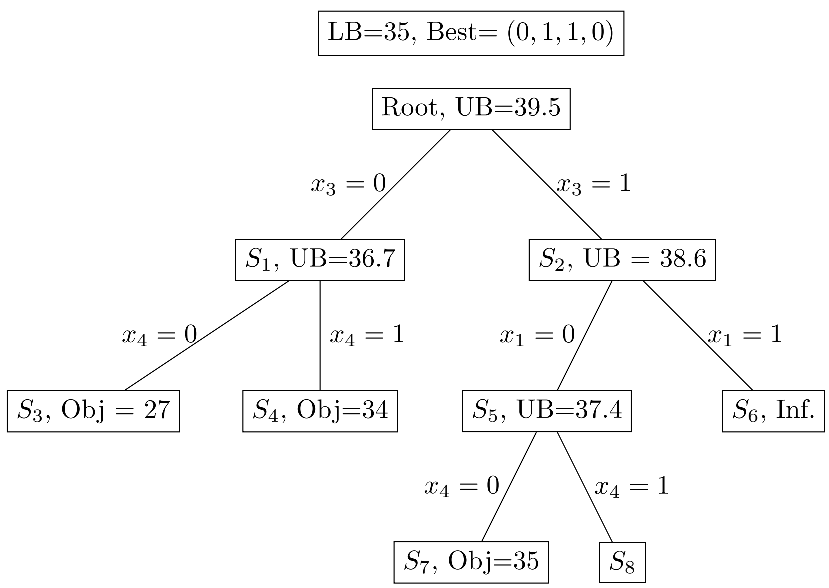
We now can evaluate node S_5 or S_6. Let's choose S_5. This gives us the solution $(0, 1, 1, 1/7)$ with objective value of 262/7 or approximately 37.4. We branch on variable x_4, giving us the tree:



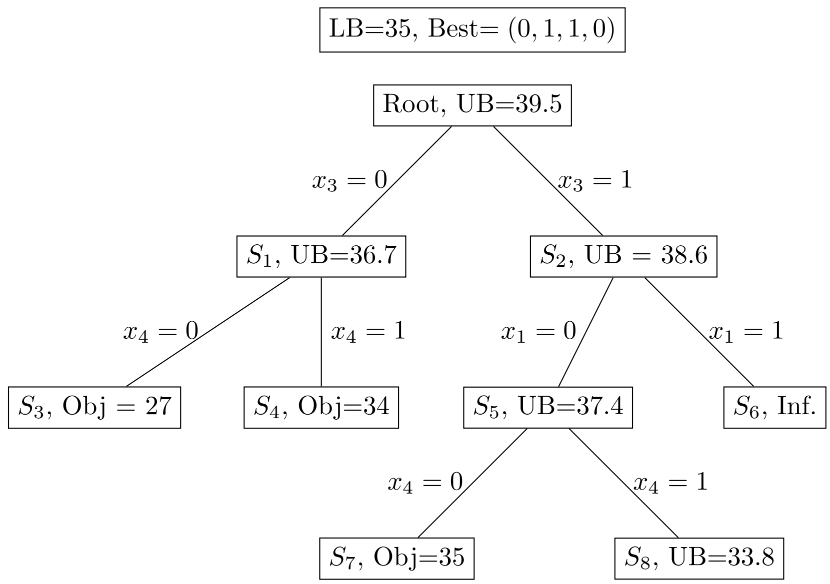
We can now evaluate S_7 or S_8. Let's evaluate S_7. This gives us the solution $(0, 1, 1, 0)$ with objective value of 35. This solution is feasible, and is better than the current best, so we update the lower bound and the best solution. There is no need to branch at this node. The tree is now:



We can now evaluate S_6 or S_8. Let's evaluate S_6. This node is infeasible, so we do not branch further on this node. The tree is now:



We now evaluate S_8. The solution at this node is $(2/5, 1, 0, 1)$, which achieves an objective value of $169/5$ or 33.8. Since our current lower bound is larger than this upper bound, there is no need to branch further at this node. The tree is now:



There are no nodes left to explore, so the best solution is $(0, 1, 1, 0)$ with an objective value of 35. □