INMAS 2021, Modeling & Optimization Problems: Session 3.

1. In the lecture, we learned that simplex starts at a vertex of the feasible region and travels to adjacent vertices. However, what do we do if we don't know any feasible solutions? It is possible to find a feasible solution by solving a different LP. Assume that we are given an LP in standard form:

$$\max c^\intercal x$$

s.t.

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Here, the vector b may contain some negative components, in which case it is not there is no obvious feasible solution. Formulate a new LP such that:

- The new LP always has a feasible solution, and you can give a simple, closed form for that solution.
- The new LP takes an objective value of zero if the original LP is feasible.
- The new LP has an objective value of greater than zero if the original LP is infeasible.
- Given a solution to the new LP with objective value of zero, you can construct a feasible solution to the original LP.

This LP can be solved to identify a feasible solution.

2. The pigeonhole principle states that the problem, "Place $n+1$ pigeons into n holes so that no two pigeons share a hole," has no solution.

   (a) Formulate this problem as an IP using the following two types of constraints:

      i. Those that enforce that every pigeon must be given a hole.
      ii. Those that enforce that, for each pair of pigeons, at most one of these pigeons can be assigned to a given hole.

   Show that the LP relaxation of this formulation is feasible.

   (b) Alternatively, formulate this problem as an IP using the following two types of constraints:

      i. Those that enforce that every pigeon must be given a hole.
      ii. Those that enforce that every hole is assigned to at most one pigeon.

   Show that the LP relaxation of this formulation is infeasible.

3. In the lecture, we saw two formulations for the same facility location problem. In fact, one is stronger than the other. Formulation A has the following constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$
$$x_{ij} \le y_j \text{ for all } i, j$$
$$x, y \ge 0$$
$$x_{ij}, y_j \in \mathbb{Z} \text{ for all } i, j$$

while Formulation B has the following constraints:

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i$$
$$\sum_{i=1}^{n} x_{ij} \le n y_j \text{ for all } j$$
$$x, y \ge 0$$
$$x_{ij}, y_j \in \mathbb{Z} \text{ for all } i, j$$

Identify which formulation is stronger than the other, and prove it. (Assume that there are at least two possible facility locations; otherwise the problem is trivial and both LP relaxations are the same.)

4. Consider the following continuous knapsack problem:

$$\max_{x} \sum_{i=1}^{n} c_i x_i$$

s.t.

$$\sum_{i=1}^{n} a_i x_i \le b$$
$$0 \le x \le 1$$

where a_i and c_i are positive numbers for each i, and b is a positive constant. Prove that the following greedy algorithm provides an optimal solution to this LP:

(a) Set $x_i = 0$ for all i. Set $r = b$; here b will represent the remaining weight. Let $I = \{1, \ldots, n\}$; here I will represent the set of remaining items.

(b) Let $t = \max_{i \in I} c_i / a_i$.

(c) Set $x_t = \min\{1, r/a_t\}$.

(d) Remove t from I.

(e) Set $r = r - a_t x_t$.

(f) If $r = 0$, then return x. Otherwise, go back to step (b).

Intuitively, this algorithm takes as much as possible of the item with the highest value-to-weight ratio until we have reached the maximum weight. Hint: form a feasible dual solution that achieves the same objective.

5. Solve the following binary knapsack problem using branch-and-bound:

$$\max_x 17x_1 + 10x_2 + 25x_3 + 17x_4$$

s.t.

$$\sum_{i=1}^{4} 5x_1 + 3x_2 + 8x_3 + 7x_4 \leq 12$$

$$x_i \in \{0, 1\}$$