INMAS 2021, Modeling & Optimization Problems: Session 2.

1. Show that the following integer programs all have the same set of feasible solutions:

   (a)

$$\begin{aligned} 97x_1 + 32x_2 + 25x_3 + 20x_4 &\leq 139 \\ x_i &\in \{0,1\} \end{aligned}$$

   (b)

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &\leq 3 \\ x_i &\in \{0,1\} \end{aligned}$$

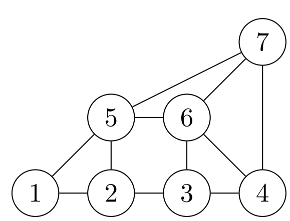
   (c)

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 2 \\ x_1 + x_2 + x_4 &\leq 2 \\ x_1 + x_3 + x_4 &\leq 2 \\ x_i &\in \{0,1\} \end{aligned}$$

2. Suppose that you are interested in choosing a set of investments $\{1, \ldots, 7\}$. Model the following constraints:

   (a) You cannot invest in all of them.
   (b) You must choose at least one of them.
   (c) Investment 1 cannot be chosen if investment 3 is chosen.
   (d) Investment 4 can be chosen only if investment 2 is also chosen.
   (e) You must choose either both investments 1 and 5 or neither.
   (f) You must choose either at least one of the investments 1,2,3 or at least two investments from 2,4,5,6.

3. A graph (V, E) is called a planar graph if it can drawn in a two-dimensional plane so that the edges E intersect only at the vertices V of the graph. A *graph coloring* is an assignment of colors to vertices such that the endpoints of each edge must be different colors. There is a theorem that states that any planar graph has a graph coloring using at most 4 colors. Formulate an integer program that identifies a coloring of a planar graph that uses the minimum number of colors. Implement this IP in Gurobi, and apply your implementation to the following graph:



4. Formulate the following as constraints to mixed integer programs:

   (a) $u = \max\{x_1, x_2\}$ assuming that $0 \leq x_j \leq C$ for $j = 1, 2$.

   (b) $v = |x_1 - x_2|$ assuming that $0 \leq x_j \leq C$ for $j = 1, 2$.

   (c) Let x be an n-dimensional vector of integer decision variables, and let $x*$ be an integral point in $\mathbb{R}^n$; model the constraint $x \neq x^*$ under the assumption that $0 \leq x_j \leq C$ for each j.

5. Suppose that you have a single machine, and you have several tasks to perform on this machine. The machine can only process one task at a time, but you may choose the order in which the tasks are performed. Each of these tasks has a deadline. Let t_j be the time required to complete task j, let d_j be the deadline for task j. There is a penalty for missing deadlines. Specifically, if task j is completed before time t_j, there is no penalty, while if task j is completed at some time $\tau > d_j$, then the penalty is $p_j\,(\tau - d_j)$ for some constant p_j.

   Hint: define variables:

$$x_j = \begin{cases} 1 & \text{the time at which job } j \text{ is completed.} \\ 0 & \text{otherwise.} \end{cases}$$

   Then, note that either $x_j \geq x_i + t_j$ or $x_i \geq x_j + t_i$; where the former holds if task i is placed before task j and the latter holds if task i is placed after task j.

   (a) Formulate an integer program that identifies the optimal sequence of tasks.

   (b) Using Gurobi, solve the following instance:

Task	Processing time	Deadline	Penalty (per minute late)
1	5 minutes	60 minutes	\$10
2	10 minutes	20 minutes	\$18
3	17 minutes	32 minutes	\$14
4	8 minutes	27 minutes	\$89
5	9 minutes	44 minutes	\$22
6	14 minutes	52 minutes	\$37
7	28 minutes	77 minutes	\$46