INMAS 2021, Modeling & Optimization: Session 1 Problem Solutions.

1. There is a discrete probability distribution that takes values $1, \ldots K$ with probabilities $p_1, \ldots p_k$. It is known that the first moment of this distribution is M_1 and the second moment of this distribution is M_2. That is,

$$\sum_{i=1}^{K} i p_i = M_1$$
$$\sum_{i=1}^{K} i^2 p_i = M_2$$

   (a) Form a linear program that identifies the probability distribution whose first and second moments match the given moments and whose fourth moment is as large as possible:

$$\sum_{i=1}^{K} i^4 p_i$$

Solution. Decision variables:

- p_i: probability of taking value i for i in $1, \ldots, K$.

Constraints:

- Probabilities must sum to one.

$$\sum_{i=1}^{n} p_i = 1$$

- First moment must be as described:

$$\sum_{i=1}^{K} i p_i = M_1$$

- Second moment must be as described:

$$\sum_{i=1}^{K} i p_i = M_2$$

Objective:

$$\max_{p} \sum_{i=1}^{K} i^4 p_i$$

□

(b) Using Gurobi, solve this linear program for $K = 5$, $M_1 = 3.5$ and $M_2 = 15$. Report the results.

Solution. See the file "fourth_moment_max.py", posted on the course website. The largest possible fourth moment is 344; a probability distribution that achieves this is:

$$P(X = k) = \begin{cases} 0.1245 & \text{for k} = 1 \\ 0.3333 & \text{for k} = 2 \\ 0 & \text{for k} = 3 \\ 0 & \text{for k} = 4 \\ 0.5417 & \text{for k} = 5 \end{cases}$$

□

(c) Using Gurobi, solve this linear program for $K = 80$, $M_1 = 10$ and $M_2 = 200$. Report the largest possible fourth moment.

Solution. See the file "fourth_moment_max.py", posted on the course website. The largest possible fourth moment is 822860.0 □

2. Given a set of points $Y = \{y_1, \dots y_k\}$ in $\mathbb{R}^n$, a point x is said to be a convex combination of Y if there exist non-negative real numbers $\alpha_1, \dots, \alpha_k$ such that

$$\sum_{i=1}^{k} \alpha_i = 1$$

and

$$\sum_{i=1}^{k} \alpha_i y_i = x.$$

The convex hull of Y is the set of all convex combinations of Y.

- Given two sets of points $Y = \{y_1, \dots, y_k\}$ and $Z = \{z_1, \dots, z_l\}$, formulate a linear program that can be used to identify whether or not the convex hull of Y intersects with that of Z.

  Solution. Decision variables:

  α_i = coefficient in convex combination of point i in Y; for $i = 1, \dots, k$.
  β_i = coefficient in convex combination of point i in Z; for $i = 1, \dots, l$.
  x_j = jth component of point in intersection of convex hulls; for $j = 1, \dots, n$.

The objective will simply be zero; the convex hulls will intersect if and only if the LP is feasible. We add constraints:

$$x_j = \sum_{i=1}^{k} \alpha_i y_{ij}$$
$$\sum_{i=1}^{k} \alpha_i = 1$$

to enforce that the vector x is a convex combination of Y. Similarly, we add constraints:

$$x_j = \sum_{i=1}^{l} \beta_i z_{ij}$$
$$\sum_{i=1}^{l} \beta_i = 1$$

to enforce that the vector x is a convex combination of Z. □

- Using Gurobi, solve this linear program for the following sets of points:

$$Y = \{(0,0), (0,2), (1,1), (2,2), (2,0)\}$$
$$Z = \{(0.5, -0.5), (3, -2), (3, 2)\}.$$

Report your results.

Solution. An implementation is available on the course website. The file is called "convex_hull_intersection.py". □

3. Consider the following linear program.

$$\max c^\mathsf{T} x$$

s.t.

$$Ax \leq b$$
$$x \geq 0$$

This form of LP is called standard form (note: standard form is not entirely standard; there are other forms that are also sometimes referred to as standard form). In fact, any LP can be written in standard form.

(a) Describe how a LP that is a minimization problem could be written as a maximization LP.

(b) Suppose that an LP has an equality constraint; i.e. a constraint of the form:

$$\alpha^\intercal x = \beta$$

where α is a vector and β is a constant. Describe how to express the equality constraint with less-than-or-equal constraints. That is, provide vectors v_1 and v_2 and constants r_1 and r_2 such that the pair

$$\begin{aligned} v_1^\intercal x &\le r_1 \\ v_2^\intercal x &\le r_2 \end{aligned}$$

is equivalent to the equality constraint.

Solution. The constraint

$$\alpha^\intercal x = \beta$$

is equivalent to:

$$\begin{aligned} \alpha^\intercal x &\le \beta \\ -\alpha^\intercal x &\le -\beta \end{aligned}$$

□

(c) Suppose that an LP has a greater-than-or-equal constraint:

$$\alpha^\intercal x \ge \beta$$

Show that this can be replaced by a less-than-or-equal constraint.

Solution. This is equivalent to the constraint:

$$-\alpha^\intercal x \le -\beta$$

□

(d) Suppose that an LP has a variable that is restricted to non-positive values; i.e. there is the constraint $x_i \le 0$ for some i. Describe how the variable x_i can then be replaced a variable that only takes non-negative values.

Solution. Define a new variable x_i' with the restriction $x_i' \ge 0$ and replace x_i with $-x_i'$ wherever it occurs. Given a solution to the new formulation, we can recover a solution to the original formulation by letting $x_i = -x_i'$. □

(e) Suppose that an LP has a variable that is not restricted to non-positive values or non-negative values; i.e. there is some i such that the LP has neither the constraint $x_i \ge 0$ nor $x_i \le 0$. Describe how x_i can be replaced by two variables that only take non-negative values.

Solution. Define a new variable x_i^+ and x_i^- with the restrictions $x_i^+ \geq 0$ and $x_i^- \geq 0$, and replace x_i with $x_i^+ - x_i^-$ wherever it occurs. The intuition here is that x_i^+ represents the positive values of x_i and x_i^- represents the negative values. Given a solution to the new formulation, we can recover a solution to the original formulation by letting $x_i = x_i^+ - x_i^-$. $\square$

Write the following linear program in standard form:

$$\min 2x_1 - x_2 + 4x_3$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_4 &\leq 2 \\ 3x_2 - x_3 &= 5 \\ x_3 + x_4 &\geq 3 \\ x_1 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

Solution. We replace the objective with:

$$\max -2x_1 + x_2 - 4x_3$$

Next, we replace the constraint:

$$3x_2 - x_3 = 5$$

with:

$$\begin{aligned} 3x_2 - x_3 &\leq 5 \\ -3x_2 + x_3 &\leq -5 \end{aligned}$$

We replace the constraint:

$$-x_3 - x_4 \leq -3$$

So far, we have:

$$\max -2x_1 + x_2 - 4x_3$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_4 &\leq 2 \\ 3x_2 - x_3 &\leq 5 \\ -3x_2 + x_3 &\leq -5 \\ x_3 + x_4 &\leq 3 \\ x_1 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

Now, we must deal with the restrictions on the variables. We introduce a new variable x_3' and replace x_3 with $-x_3'$:

$$\max -2x_1 + x_2 + 4x_3'$$

s.t.

$$\begin{aligned}
x_1 + x_2 + x_4 &\leq 2 \\
3x_2 + x_3' &\leq 5 \\
-3x_2 - x_3' &\leq -5 \\
-x_3' + x_4 &\leq 3 \\
x_1 &\geq 0 \\
x_3' &\geq 0
\end{aligned}$$

We introduce variables x_2^+ and x_2^- and replace x_2 with $x_2^+ - x_2^-$.

$$\max -2x_1 + (x_2^+ - x_2^-) + 4x_3'$$

s.t.

$$\begin{aligned}
x_1 + (x_2^+ - x_2^-) + x_4 &\leq 2 \\
3(x_2^+ - x_2^-) + x_3' &\leq 5 \\
-3(x_2^+ - x_2^-) - x_3' &\leq -5 \\
-x_3' + x_4 &\leq 3 \\
x_1 &\geq 0 \\
x_2^+ &\geq 0 \\
x_2^- &\geq 0 \\
x_3' &\geq 0
\end{aligned}$$

We introduce variables x_4^+ and x_4^- and replace x_4 with $x_4^+ - x_4^-$.

$$\max -2x_1 + (x_2^+ - x_2^-) + 4x_3'$$

s.t.

$$
\begin{aligned}
x_1 + (x_2^+ - x_2^-) + (x_4^+ - x_4^-) &\le 2 \\
3(x_2^+ - x_2^-) + x_3' &\le 5 \\
-3(x_2^+ - x_2^-) - x_3' &\le -5 \\
-x_3' + (x_4^+ - x_4^-) &\le 3 \\
x_1 &\ge 0 \\
x_2^+ &\ge 0 \\
x_2^- &\ge 0 \\
x_3' &\ge 0 \\
x_4^+ &\ge 0 \\
x_4^- &\ge 0
\end{aligned}
$$

The LP is now in standard form. □

4. Consider the problem:

$$\min_x \frac{c^\intercal x + d}{f^\intercal x + g}$$

subject to:

$$
\begin{aligned}
Ax &\le b \\
x &\ge 0 \\
f^\intercal x + g &> 0
\end{aligned}
$$

where c and f are vectors, and d and g are constants. Suppose that we know that the optimal value of this problem is at least ℓ and at most u. Describe a procedure that takes a specified value ϵ and finds a solution that is with ϵ of the optimal solution. Assume in your procedure that you have an oracle that can solve an LP if it is feasible, or identify that it is infeasible. Hint: first, consider how you could check whether or not the optimal objective value is less than or equal to some number. You may assume that there does not exist any solution in which $Ax \le b$, $x \ge 0$, and $f^\intercal x + g = 0$ (if you would like an extra challenge, you can relax this assumption).

Solution. We note that if the objective is bounded, First, we note that the optimal objective value of the given problem is in some interval $[\alpha, \beta]$ if and only if there exists some x such that

$$
\begin{aligned}
\frac{c^\intercal x + d}{f^\intercal x + g} &\le z \\
c^\intercal x + d &\le z(f^\intercal x + g)
\end{aligned}
$$

Thus, the optimal objective value is less than z if and only there exists a feasible solution to the constraints:

$$\begin{aligned} c^\intercal x + d &\leq z(f^\intercal x + g) \\ Ax &\leq b \\ x &\geq 0 \\ f^\intercal x + g &\geq 0 \end{aligned}$$

We can solve an LP with objective value of zero to identify if this is feasible. Let $P(z)$ refer to the LP with bound z on the objective value.

One possible algorithm is as follows:

(a) Set $L = \ell$ and $U = u$. The variables L and U will store a lower bound and an upper bound on the objective value.

(b) Use a linear programming solver to solve $P(U)$. Since the objective value can be no more than U, this must result in a solution. Let x^* be the solution.

(c) Set $z = (L + U)/2$.

(d) Use a linear programming solver to solve $P(z)$. If this is feasible, set x^* to the solution, and set $U = z$. If this is infeasible, set $L = z$.

(e) If $U - L < \epsilon$, return x^*. Otherwise, go back to step (c).

□

5. A function f is called *piecewise linear convex* if there is a finite set V of pairs of vectors (α, β) such that

$$f(x) = \max_{(\alpha,\beta)\in V} \alpha^\intercal x + \beta$$

Suppose that you have an optimization problem of the form:

$$\min_x f(x)$$

s.t.

$$\begin{aligned} g_i(x) &\leq 0 \text{ for } i \in \{1, \ldots, k\} \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

where f and $g_1, \ldots, g_k$ are piecewise linear convex functions. Provide a linear program formulation that can be used to solve this optimization problem.

Solution. If we introduce a single variable θ, then we can rewrite the problem as:

$$\min_{\theta,x} \theta$$

s.t.

$$\begin{aligned} \theta &= f(x) \\ g_i(x) &\leq 0 \text{ for } i \in \{1, \ldots, k\} \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

We will refer to this formulation as P_1. We then note that we can relax this optimization problem to:

$$\min_{\theta,x} \theta$$

s.t.

$$\begin{aligned} \theta &\geq f(x) \\ g_i(x) &\leq 0 \text{ for } i \in \{1, \ldots, k\} \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

We will refer to this formulation as P_2. We claim that any optimal solution to P_2 will be an optimal solution to P_1. First, we show that any optimal solution to P_2 is feasible for P_1. Suppose that we have an optimal solution θ^* and x^* to P_2, and suppose that $f(x^*) < \theta^*$. In this case, we could let $\theta' = f(x^*)$, and the solution (θ', x^*) would have a lower objective value than that of (θ^*, x^*). This implies that $\theta = f(x)$ in an optimal solution to P_2. Next, we claim that there is no feasible solution to P_1 that achieves a better objective value than the optimal solution to P_2. This must be true because every feasible to solution to P_1 is also a feasible to solution to P_2.

Next, we note that we can rewrite the constraint $\theta \geq f(x)$ as

$$f(x) - \theta \leq 0$$

We further note that since f is piecewise linear convex, then the function $f(\cdot) - \theta$ is also piecewise-linear convex. This is true because

$$\left(\max_{(\alpha,\beta)\in V} \alpha^\intercal x + \beta \right) - \theta = \max_{(\alpha,\beta)\in V} \alpha^\intercal x + (\beta - \theta)$$

So, this whole problem can be reduced to rewriting a constraint of the form

$$h(x) \leq 0$$

as one or more linear constraints when h is piecewise linear. Next, we note that

$$\max_{(\alpha,\beta)\in V} \alpha^\intercal x + \beta \leq 0$$

if and only if

$$\alpha^\intercal x + \beta \leq 0 \text{ for all } (\alpha, \beta) \in V$$

which is a finite set of linear constraints. Then, if we apply this transformation to the constraint

$$f(x) - \theta \leq 0$$

and to each of the constraints

$$g_i(x) \leq 0$$

we will achieve a LP formulation. □