INMAS 2021, Modeling & Optimization: Session 1 Problems.

1. There is a discrete probability distribution that takes values $1, \dots K$ with probabilities $p_1, \dots p_k$. It is known that the first moment of this distribution is M_1 and the second moment of this distribution is M_2. That is,

$$\sum_{i=1}^{K} i p_i = M_1$$

$$\sum_{i=1}^{K} i^2 p_i = M_2$$

   (a) Form a linear program that identifies the probability distribution whose first and second moments match the given moments and whose fourth moment is as large as possible:

$$\sum_{i=1}^{K} i^4 p_i$$

   (b) Using Gurobi, solve this linear program for $K = 5$, $M_1 = 3.5$ and $M_2 = 15$. Report the results.

   (c) Using Gurobi, solve this linear program for $K = 80$, $M_1 = 10$ and $M_2 = 200$. Report the largest possible fourth moment.

2. Given a set of points $Y = \{y_1, \dots y_k\}$ in $\mathbb{R}^n$, a point x is said to be a convex combination of Y if there exist non-negative real numbers $\alpha_1, \dots, \alpha_k$ such that

$$\sum_{i=1}^{k} \alpha_i = 1$$

and

$$\sum_{i=1}^{k} \alpha_i y_i = x.$$

The convex hull of Y is the set of all convex combinations of Y.

   - Given two sets of points $Y = \{y_1, \dots, y_k\}$ and $Z = \{z_1, \dots, z_l\}$, formulate a linear program that can be used to identify whether or not the convex hull of Y intersects with that of Z.
   - Using Gurobi, solve this linear program for the following sets of points:

$$Y = \{(0,0), (0,2), (1,1), (2,2), (2,0)\}$$
$$Z = \{(0.5, -0.5), (3, -2), (3, 2)\}.$$

   Report your results.

3. Consider the following linear program.

$$\max c^\intercal x$$

s.t.

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \end{aligned}$$

This form of LP is called standard form (note: standard form is not entirely standard; there are other forms that are also sometimes referred to as standard form). In fact, any LP can be written in standard form.

(a) Describe how a LP that is a minimization problem could be written as a maximization LP.

(b) Suppose that an LP has an equality constraint; i.e. a constraint of the form:

$$\alpha^\intercal x = \beta$$

where α is a vector and β is a constant. Describe how to express the equality constraint with less-than-or-equal constraints. That is, provide vectors v_1 and v_2 and constants r_1 and r_2 such that the pair

$$\begin{aligned} v_1^\intercal x &\leq r_1 \\ v_2^\intercal x &\leq r_2 \end{aligned}$$

is equivalent to the equality constraint.

(c) Suppose that an LP has a greater-than-or-equal constraint:

$$\alpha^\intercal x \geq \beta$$

Show that this can be replaced by a less-than-or-equal constraint.

(d) Suppose that an LP has a variable that is restricted to non-positive values; i.e. there is the constraint $x_i \leq 0$ for some i. Describe how the variable x_i can then be replaced a variable that only takes non-negative values.

(e) Suppose that an LP has a variable that is not restricted to non-positive values or non-negative values; i.e. there is some i such that the LP has neither the constraint $x_i \geq 0$ nor $x_i \leq 0$. Describe how x_i can be replaced by two variables that only take non-negative values.

Write the following linear program in standard form:

$$\min 2x_1 - x_2 + 4x_3$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_4 &\leq 2 \\ 3x_2 - x_3 &= 5 \\ x_3 + x_4 &\geq 3 \\ x_1 &\geq 0 \\ x_3 &\leq 0 \end{aligned}$$

4. Consider the problem:

$$\min_x \frac{c^\intercal x + d}{f^\intercal x + g}$$

subject to:

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \\ f^\intercal x + g &> 0 \end{aligned}$$

where c and f are vectors, and d and g are constants. Suppose that we know that the optimal value of this problem is at least ℓ and at most u. Describe a procedure that takes a specified value ϵ and finds a solution that is with ϵ of the optimal solution. Assume in your procedure that you have an oracle that can solve an LP if it is feasible, or identify that it is infeasible. Hint: first, consider how you could check whether or not the optimal objective value is less than or equal to some number. You may assume that there does not exist any solution in which $Ax \leq b$, $x \geq 0$, and $f^\intercal x + g = 0$ (if you would like an extra challenge, you can relax this assumption).

5. A function f is called *piecewise linear convex* if there is a finite set V of pairs of vectors (α, β) such that

$$f(x) = \max_{(\alpha,\beta)\in V} \alpha^\intercal x + \beta$$

Suppose that you have an optimization problem of the form:

$$\min_x f(x)$$

s.t.

$$\begin{aligned} g_i(x) \leq 0 &\text{ for } i \in \{1, \ldots, k\} \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

where f and $g_1, \ldots, g_k$ are piecewise linear convex functions. Provide a linear program formulation that can be used to solve this optimization problem.