SUPPLEMENTARY MATERIALS: FACETS OF THE STOCHASTIC NETWORK FLOW PROBLEM

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SM1. Proof of Lemma 4.1.

Statement of Lemma 4.1. Let H be an acyclic F-graph. Then H has disjoint reachability if and only if $\{U(w;H) : w \in W\}$ is a collection of disjoint sets for every edge $e = (v, W)$ in $E(H)$.

Proof of Lemma 4.1. First we prove the forward direction, that if H has disjoint reachability then $\{U(w;H) : w \in W\}$ is a collection of disjoint sets for every edge $e = (v, W)$ in $E(H)$. In fact, we prove the contrapositive. Assume that there is some edge $e = (v, W)$ where $\{U(w, H) : w \in W\}$ is not a collection of disjoint sets. Then we can produce a compath that is not an out-tree. By the assumption, there exists some $\rho \in V(H)$ and some distinct $w_1, w_2 \in W$ such that $\rho \succeq^H w_1$ and $\rho \succeq^H w_2$. In particular, let ρ be a minimal node such that $\rho \succeq^H w_1$ and $\rho \succeq^H w_2$. Let P_1 be the path from w_1 to ρ and let P_2 be the path from w_2 to ρ. We claim that $P^* = e + P_1 + P_2$ is a compath. It is clear that v is the unique minimal node of P^*. Next, we need to show that each node has at most one outgoing edge. Suppose that some node $u \in P^*$ has two outgoing edges e_1 and e_2. Since the paths P_1 and P_2 start at nodes strictly greater than v, it is clear that $u \neq v$. Consequently, $e \neq e_1$ and $e \neq e_2$, so these edges must have come from P_1 and P_2. In any path each node has only one outgoing edge, so one of the two edges is in P_1 while the other edge is in P_2. This implies that $u \succeq^H w_1$ and $u \succeq^H w_2$, since w_1 and w_2 are the first nodes in the respective paths. Since ρ is the final node of the path P_1, then $u \preceq^P \rho$. We defined ρ to be a minimal node such that $\rho \succeq^H w_1$ and $\rho \succeq^H w_2$, so in fact it must be true that $u = \rho$. However, ρ has no outgoing edges in the path P_1 or P_2. This is a contradiction. Thus, there is no node $u \in P^*$ that has two outgoing edges e_1 and e_2. Thus, P is a compath. Next, we show that ρ has two incoming edges in the compath P. Let η_1 be the incoming edge of ρ in P_1 and let η_2 be the incoming edge of ρ in P_2. Suppose that $\eta_1 = \eta_2$, and let y be the origin node of this edge. Then, it would be true that $y \succeq^H w_1$ and $y \succeq^H w_2$ since w_1 and w_2 are the first nodes in their respective paths. It would also be true that $y \prec^H \rho$ since there is a path from y to ρ consisting of the single edge η_1. However, this is a contradiction since by definition, ρ is a minimal node such that $\rho \succeq^H w_1$ and $\rho \succeq^H w_2$. Thus, the node v has two incoming edges in the compath P. We have now constructed a compath P of H that is not an out-tree, which completes the proof of the forward direction.

In the opposite direction, suppose that H does not have disjoint reachability. Then there exists some compath P such that some node $u \in V(P)$ has two distinct incoming edges $e_1 = (v_1, W_1)$ and $e_2 = (v_2, W_2)$ in P. Let v_0 be the starting node of compath P. Let z be a maximal node preceding both v_1 and v_2 in P. That is, let z be such that $z \prec^P v_1$ and $z \prec^P v_2$ and such that if y is another node where $y \prec^P v_1$ and $y \prec^P v_2$ then $z \not\prec^P y$. The existence of such a node is guaranteed by the fact that $v_0 \prec^P v_1$ and $v_0 \prec^P v_2$. By definition of a compath, there are paths P_1 and P_2 in P starting at z with destinations v_1 and v_2 respectively. Let w_1 and w_2 be the nodes

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immediately following z in the paths P_1 and P_2 respectively. Since P is a compath, z has only one outgoing edge η in P. Since P_1 and P_2 are subgraphs of P, then w_1 and w_2 must both be destination nodes of η. By the fact that $w_1 \prec^P v_1$ and $w_2 \prec^P v_2$, it must be true that w_1 is distinct from w_2. Otherwise, it would be true that $z \prec^P w_1$, $w_1 \prec^P v_1$, and $w_1 \prec^P v_2$; this contradicts the definition of z. Now, since $v_1 \prec^H u$ and $w_1 \prec^H v_1$, we see that $w_1 \prec^H u$. Similarly, $w_2 \prec^H u$. Thus, both the set $U(w_1; H)$ and $U(w_2; H)$ contain the node u. Then, $\{U(w; H) : w \in W\}$ is not a collection of disjoint sets when W is taken to be the destination nodes of η. ∎

SM2. Notation. Here, we define notation that is used in several sections of the online supplement. Let $\boldsymbol{\lambda}$ be a solution to a F-graph flow problem on some F-graph H. We will let $f_e(\boldsymbol{\lambda})$ be the value of the variable f_e in the solution $\boldsymbol{\lambda}$ for each edge e in $E(H)$. We will use the notation

$$F^+(\boldsymbol{\lambda}) := \{e \in E(H) : f_e(\boldsymbol{\lambda}) > 0\},$$

the set of edges that take positive flow in the solution $\boldsymbol{\lambda}$. Similarly, we define

$$F^0(\boldsymbol{\lambda}) := \{e \in E(H) : f_e(\boldsymbol{\lambda}) = 0\},$$

the set of edges that take zero flow in the solution $\boldsymbol{\lambda}$. For a set of edges $\mathcal{E}$ we let $\boldsymbol{\ell}(\mathcal{E})$ be the solution (not usually feasible) defined by

$$f_e(\boldsymbol{\ell}(\mathcal{E})) = \begin{cases} 1 & e \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

For a subgraph S, we use the shorthand $\boldsymbol{\ell}(S) = \boldsymbol{\ell}(E(S))$.

SM3. Constructing Feasible Solutions. We present several lemmas that assist in the construction and modification of feasible solutions. These lemmas will be useful in the proof of subsequent results. In this section, we use the notation described in section SM2.

Lemma SM3.1 states that the subgraph induced by the edges greater than some node v in a spanning in-forest X is always a maximal compath.

LEMMA SM3.1. *Let X be a spanning in-forest of an ADF-hypergraph H, and let v be any node in $V(H)$. Then $U(v; X)$ is a maximal compath.*

Proof. The subnetwork $U(v; X)$ is clearly a compath since v is the unique minimal node of $U(v; X)$, and every node has exactly one outgoing edge in X. We can also show that it is maximal. Let $e^* = (x, W)$ be a maximal edge of $E(U(v; X))$ in X and suppose that e^* is not a maximal edge in H. In this case, there exists some edge $\tilde{e} = (x', W')$ in $E(H)$ such that there is a path P from some node w in W to x'. However, since $x' \in V(H)$ and X is a spanning in-forest, then there exists some edge e' in $E(X)$ whose origin node is w'. Then, $e' \succ^X e^*$, which is a contradiction because e^* is a maximal edge in X. Thus, $U(v; X)$ is a maximal compath. ∎

Lemma SM3.2 allows the creation of larger compaths from smaller compaths.

LEMMA SM3.2. *Let $e = (v, W)$ be any edge of an ADF-graph H. For each $w \in V(H) \cap W$, let P_w be a compath beginning at w. Then the subgraph $P^* = e + \sum_{w \in V(H) \cap W} P_w$ is a compath. Furthermore, P^* is maximal if and only if each P_w is maximal.*

Proof. It is easy to see that v is the unique minimal node of P^*. It follows from Lemma 4.1 that the paths P_w for $w \in V \cap W$ cannot have any edges or vertices in common, which in turn implies that each node of P^* has at most one outgoing edge. It is also clear that an edge is a maximal edge of P^* if and only if it is a maximal edge of P_w for some $w \in V(H) \cap W$. This immediately implies that P^* is maximal if and only if each P_w is maximal. ∎

Lemma SM3.3 states that there is always a compath contained in the edges that take positive flow in a feasible solution to a flow problem.

LEMMA SM3.3. *Let $\boldsymbol{\phi}$ be any feasible solution to an F-graph flow problem on some ADF-graph H, and let $e = (v, W)$ be an edge of $F^+(\boldsymbol{\phi})$. Then there exists a maximal compath P in $F^+(\boldsymbol{\phi})$ starting at v whose first edge is e.*

Proof. Induction. This is clearly true for any maximal edge e, because the edge itself is a maximal compath. Suppose that there exists some edge $e = (v, W)$ in $F^+(\boldsymbol{\phi})$ such that the property is true for all edges $e' \succ e$; we will show that the property is also true for e. If $W \cap V(H)$ is empty, then the edge is maximal and the property is again satisfied. Suppose $W \cap V(H)$ is not empty. Since there is positive flow on the edge e, then for each vertex $w \in W \cap V(H)$ there must exist some edge η_w outgoing from w such that η_w has positive flow; otherwise, the conservation constraint would not be satisfied at w. By the induction hypothesis, there is a maximal compath P_w starting at w. Then, by Lemma SM3.2, $e + \sum_{w \in V(H) \cap W} P_w$ is a maximal compath. ∎

Lemma SM3.4 states that adding a unit of flow along compath and while removing a unit of flow along a compath that starts at the same node preserves feasibility, as long as edge capacities are not violated.

LEMMA SM3.4. *Let $\boldsymbol{\phi}$ be an integral feasible solution to an F-graph flow problem on some ADF-graph H. Let P' be any maximal compath in $F^+(\boldsymbol{\phi})$ starting at some node v of $V(H)$. Let P be any maximal compath starting at the same node v such that*

$$f_e(\boldsymbol{\phi}) \leq a_e - 1$$

for any edge e in $(E(P) \cap \mathscr{C}) \setminus E(P')$. Then $\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')$ is an integral feasible solution. Furthermore, $E(\boldsymbol{\ell}(P)) \subseteq F^+(\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P'))$.

Proof. Note that the variables in $\boldsymbol{\ell}(P)$ and $\boldsymbol{\ell}(P')$ only appear in the conservation constraints for nodes that fall in the paths P and P', the edge capacity constraints for edges in these paths, and the non-negativity constraints for edges in these paths. Since P' is a compath within the edges of $\boldsymbol{\phi}$ that have positive flow and the solution is integral, then all variables of $\boldsymbol{\phi} - \boldsymbol{\ell}(P')$ take positive values, which in turn implies that all variables of $\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')$ take positive values.

The node conservation constraint at v will still be satisfied because the addition of $\boldsymbol{\ell}(P)$ increases the outgoing flow by one, while the subtraction of $\boldsymbol{\ell}(P')$ decreases this flow by one, and neither of the paths alter the incoming flow. For any node w other than v, note that since P' and P are out-trees and maximal, the node w has exactly one outgoing and one incoming arc, so the addition of $\boldsymbol{\ell}(P)$ and the subtraction of $\boldsymbol{\ell}(P')$ do not violate the conservation constraint.

Observe that

$$f_e(\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')) = \begin{cases} f_e(\boldsymbol{\phi}) + 1 & e \in E(P) \setminus E(P'), \\ f_e(\boldsymbol{\phi}) & e \in E(P') \cap E(P), \\ f_e(\boldsymbol{\phi}) - 1 & e \in E(P') \setminus E(P). \end{cases}$$

From this and the assumptions of the lemma it is apparent that the edge capacity constraints are preserved. It follows from this same observation that $E(\boldsymbol{\ell}(P)) \subseteq F^+(\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P'))$. Let e be some edge of the path P. From the observation, it is immediate that e receives positive flow in $\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')$ if e is not in P'. If e is in P', then

$$f_e(\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')) = f_e(\boldsymbol{\phi})$$

and, by the assumptions of the lemma, $f_e(\boldsymbol{\phi}) \geq 1$. Thus, in either case, the edge e receives positive flow in $\boldsymbol{\phi} + \boldsymbol{\ell}(P) - \boldsymbol{\ell}(P')$. ∎

Given a path P and a spanning in-forest X, it is possible to extend the path into a maximal compath using only edges from the spanning in-forest.

LEMMA SM3.5. *Let P be a compath in an ADF-graph H starting at some node v in $V(H)$, and let X be a spanning in-forest of H. There exists a maximal compath P' of H starting at v such that $E(P) \subseteq E(P') \subseteq E(P) \cup X$.*

Proof. We prove this by induction on the number of edges in P. If P consists of a single edge $e = (v, W)$, then it follows from Lemma SM3.1 and Lemma SM3.2 that

$$P' = e + \sum_{w \in W} U(w; X)$$

is a maximal compath. Suppose that this theorem holds for any compath with k edges, and let P be a compath with $k+1$ edges. Let $e = (v, W)$ be the outgoing edge of v in P. Consider $U(w; P)$ for some w in $W \cap V(H)$. If w has an outgoing edge in P then $U(w; P)$ is a compath, and this compath cannot include the edge e, so it has at most k edges. By induction, there exist a maximal compath P_w starting at w such that $U(w; P) \subseteq P_w$ and $E(P_w) \subseteq E(U(w; P)) \cup X$. If w has no outgoing edge in P, then let $P_w = U(w; X)$, which is a maximal compath by Lemma SM3.1. Then, we claim that

$$P' = e + \sum_{w \in W \cap V(H)} P_w$$

is a maximal compath that satisfies the desired properties. Lemma SM3.2 guarantees that this is a maximal compath starting at v.

By definition of each compath P_w, $E(P_w) \subseteq E(U(w; P)) \cup X$, so it must be true that

$$E(P') \subseteq e + \left(\bigcup_{w \in W} E(U(w; P)) \right) \cup X.$$

From this, any edge η in P' is either the edge e, is in the subgraph $U(w; P)$ for some $w \in W \cap V(H)$, or is an edge of X. Thus, any edge of P' must be an edge of $E(P) \cup X$.

Let η be an edge of P. If $\eta = e$, then it is clear that η is an edge of P'. If $\eta \neq e$, then it must be true that $\eta \succ^P w$ for some destination node w of e. This implies that w has an outgoing edge in P and that η is in $U(w; P)$. By definition, $P_w = U(w; P)$ when w has an outgoing edge in P, so η is in P_w. Thus, $E(P) \subseteq E(P')$. ∎

SM4. Proof of Proposition 5.1.

Statement of Proposition 5.1. If an F-graph has a loose cycle with an odd number of sideways edges, then the corresponding flow problem does not have a TU constraint matrix.

Proof of Proposition 5.1. Let $(e_1, ..., e_k)$ be a loose cycle in some F-graph. For convenience, throughout this proof we will use the convention that $k+1$ is equal to 1. Let v_i be the single node shared by e_i and e_{i+1} for each i between 1 and k. Consider the submatrix M defined by the flow variables for edges $e_1, ..., e_k$ and the conservation constraints corresponding to nodes $v_1, ..., v_k$. We will assume that the entries of M are arranged so that each entry M_{ij} corresponds to the conservation of flow constraint for node v_i and the flow variable on edge e_j. We claim this matrix is not TU. If M were TU, then it would be possible to assign a value s_i in $\{-1, +1\}$ for each i between 1 and k such that

$$\sum_{i=1}^{k} s_i M_{ij} \in \{-1, 0, 1\}$$

for each j between 1 and k. Suppose that such a vector $\mathbf{s}$ exists. Note that the column corresponding to some edge e_i has exactly two non-zero entries. More specifically, these non-zero entries appear in rows corresponding to the nodes v_i and v_{i+1}. If e_i is a sideways edge, then both of these nodes are destination nodes of e_i, and the corresponding non-zero elements in the matrix M will both take the value -1. Otherwise, one of these entries will take the value 1 and the other will take the value -1. From this, we can see that it must be true that

$$s_{i+1} = \begin{cases} -s_i & \text{if } e_i \text{ is a sideways edge} \\ s_i & \text{otherwise} \end{cases}$$

for each i between 1 and k. This implies that

$$s_{k+1} = (-1)^m s_1$$

where m is the number of sideways edges in the cycle. Since m is odd,

$$s_{k+1} = -s_1,$$

but by definition

$$s_{k+1} = s_1.$$

∎

SM5. Proof of Proposition 5.3.

Statement of Proposition 5.3. Let $\mathscr{P}(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, \mathbf{c})$ be the system of constraints for an F-graph flow problem that satisfies the following:

1. H has at least one delivery edge;
2. $\mathscr{C}$ is empty.

This system of constraints is a Leontief substitution system.

Proof. In order to be a Leontief substitution system, the right-hand-side vector must be non-negative. In this case, the elements of the right-hand-side vector of the min-cost problem are the quantities supplied to the nodes, which we restrict to take non-negative values.

We must also show that the constraint matrix is a Leontief matrix. Each column of the constraint matrix corresponds to some edge e, and has exactly one positive entry corresponding to the origin node of e. Thus, the constraint matrix satisfies the condition that every column must have exactly one positive entry. This relies on the assumption that there are no uncapacitated edges, since the column of the constraint matrix corresponding to a capacitated edge e has two positive elements: one corresponding to the origin node of e and one corresponding to the capacity constraint. We can construct a non-negative vector x^* such that $Ax^* > 0$. By assumption, H has at least one delivery edge η. The column of the constraint matrix corresponding to this delivery edge has one positive element and has no negative elements. Using this fact, simply let the element of x^* corresponding to η be equal to one, and let all other elements be equal to zero. ∎

SM6. Proof of Lemma 5.4. In this section, we use the notation defined in section SM2.

Statement of Lemma 5.4. Let $\boldsymbol{q}, \mathscr{C}$, and $\mathbf{a}$ be parameters defining the polytope $\mathscr{P}_1(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a})$ of a flow problem on H, and suppose that there exists a spanning in-forest X that consists of edges in $E(H) \setminus \mathscr{C}$. Let $\boldsymbol{\phi}$ be an (integral) feasible solution in $\mathscr{P}_1(S, \boldsymbol{q}, \mathscr{C}, \mathbf{a})$ for some subnetwork S of H. Then there exists an (integral) feasible solution $\boldsymbol{\phi}'$ in $\mathscr{P}_1(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a})$ such that $f_e(\boldsymbol{\phi}) = f_e(\boldsymbol{\phi}')$ for all edges $e \in E(S) \setminus X$ and such that $f_e(\boldsymbol{\phi}') = 0$ for all edges $e \in E(H) \setminus (X \cup E(S))$.

Proof of Lemma 5.4. For each node $v \in V(H)$, let

$$R_v = q_v + \sum_{e \in E^-(v,S)} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(v,S)} f_e(\boldsymbol{\phi}).$$

For any node $v \in V(S)$, $R_v = 0$ from the conservation constraints in $P_1(S, \boldsymbol{q}, \mathscr{C}, \mathbf{a})$. Note that no node v from $V(H) \setminus V(S)$ can have outgoing arcs in S. Thus, $R_v \geq 0$ for all v. We claim that the solution

$$\boldsymbol{\phi}' := \boldsymbol{\phi} + \sum_{v \in V(H)} R_v \boldsymbol{\ell}(U(v; X))$$

has the desired properties. Since $R_v = 0$ for any node v in $V(S)$ then $f_e(\boldsymbol{\phi}) = f_e(\boldsymbol{\phi}')$ for all edges $e \in E(S) \setminus X$. It is also easy to see that any edge capacity constraints will be satisfied by $\boldsymbol{\phi}'$ since it takes the same values as $\boldsymbol{\phi}$ in all capacitated edges.

Note that for any edge $e \in X$,

$$f_e(\boldsymbol{\phi}') := f_e(\boldsymbol{\phi}) + \sum_{v \in V(L(e;X))} R_v.$$

Consider some node $v \in V(H)$;

$$\begin{aligned} q_v + \sum_{e \in E^-(v,H)} f_e(\boldsymbol{\phi}') - \sum_{e \in E^+(v,H)} f_e(\boldsymbol{\phi}') \\ &= q_v + \sum_{e \in E^-(v,S)} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(v,S)} f_e(\boldsymbol{\phi}) \\ &+ \sum_{e \in E^-(v,X)} \sum_{w \in V(L(e;X))} R_w - \sum_{\substack{w \in V(H): \\ w \prec^X e^+(v;X)}} R_w. \end{aligned}$$

We claim that the collection $\{V(L(e;X)); e \in E^-(v;X)\}$ is a partition of $V(L(v;X)) \setminus \{v\}$. It is straightforward to verify that the union of this collection is equal to $V(L(v;X)) \setminus \{v\}$. Suppose that these sets are not disjoint, so that there is some node w in $V(L(\eta_1;X))$ and in $V(L(\eta_2;X))$ for distinct edges η_1 and η_2 in $E^-(v;X)$. By Lemma SM3.1, $U(w;X)$ is a compath. Since H is an ADF-graph, this compath is an out-tree, which implies that v has at most one incoming edge in $U(w;X)$. However, both η_1 and η_2 are in $U(w;X)$, and both are incoming edges of v. This is a contradiction. This shows that $\{V(L(e;X)); e \in E^-(v;X)\}$ is a partition of $V(L(v;X)) \setminus \{v\}$. Using this fact, the right-hand-side can be rewritten as

$$\begin{aligned}\text{RHS} &= q_v + \sum_{e \in E^-(v,S)} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(v,S)} f_e(\boldsymbol{\phi}) + \sum_{\substack{w \in V(H): \\ w \prec^X v}} R_w - \sum_{\substack{w \in V(H): \\ w \preceq^X v}} R_w \\ &= q_v + \sum_{e \in E^-(v,S)} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(v,S)} f_e(\boldsymbol{\phi}) - R_v \\ &= 0.\end{aligned}$$

This demonstrates that $\boldsymbol{\phi}'$ is feasible. Further note that if $\boldsymbol{\phi}$ is integral, then each R_v is an integer. Thus, $\boldsymbol{\phi}'$ will also be integral. ∎

SM7. Proof of Theorem 6.1.

Statement of Theorem 6.1. Let there be an ADF-graph H, a set of edges $\mathscr{C}$ in $E(H)$, and a spanning in-forest X of H such that X is disjoint from $\mathscr{C}$. For any vector of supplies $\mathbf{q}$ and capacities $\mathbf{a}$, $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a}) = \mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$.

Proof. Proof of Theorem 6.1 Define the *pseudo-rank* $r(v;X)$ of a node v to be the number of edges in the shortest path from any minimal node to v under the partial ordering $\prec^X$, with the convention that $r(v;X) = 0$ if v is a minimal node. Let $V^n(H;X)$ be the nodes that have pseudo-rank of at most n in the poset on $V(H)$ with ordering $\prec^X$. We define $\mathscr{P}_1^n(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$ to be the polytope in $\mathbb{R}^{|E|}$ formed by the conservation constraints for the nodes of $V^n(H;X)$ along with the capacity and non-negativity constraints on the outgoing edges on these nodes. That is,

$$\mathscr{P}_1^n(H, \mathbf{q}, \mathscr{C}, \mathbf{a}) := \boldsymbol{f} \in \mathbb{R}^{|E|} \text{ such that}$$

$$\begin{aligned}\sum_{e \in E^+(v;H)} f_e = q_v + \sum_{e \in E^-(v;H)} f_e \quad & \forall v \in V^n(H;X), \\ f_e \le a_e \quad & \forall e \in \mathscr{C} \text{ s.t. the origin of } e \text{ is in } V^n(H;X), \\ f_e \ge 0 \quad & \forall e \in E(H) \text{ s.t. the origin of } e \text{ is in } V^n(H;X).\end{aligned}$$

We define $\mathscr{P}_2^n(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$ similarly. Since the parameters of these polytopes will remain constant throughout this proof, we will refer to these polytopes simply as $\mathscr{P}_1^n$ and $\mathscr{P}_2^n$.

We prove this property using induction on n. The nodes with pseudo-rank 0 are the minimal nodes of the poset. Note that the capacity and non-negative constraints are always the same in $\mathscr{P}_1$ and $\mathscr{P}_2$ and that for each minimal node v the conservation constraint is the same in $\mathscr{P}_1$ as it is in $\mathscr{P}_2$. Thus, $\mathscr{P}_1^0 = \mathscr{P}_2^0$. Suppose that $\mathscr{P}_1^{n-1} = \mathscr{P}_2^{n-1}$ for some $n \ge 1$ and consider some feasible solution $\boldsymbol{\phi}$ of $\mathscr{P}_1^n$. For brevity, we will use the notation:

$$R(w; H, X) = q_w + \sum_{e \in E^-(w;H) \setminus X} \boldsymbol{\phi}_e - \sum_{e \in E^+(w;H) \setminus X} \boldsymbol{\phi}_e.$$

By definition $\mathscr{P}_1^n \subseteq \mathscr{P}_1^{n-1}$ and by the induction assumption $\mathscr{P}_1^{n-1} = \mathscr{P}_2^{n-1}$. Thus, $\boldsymbol{\phi}$ is a feasible solution of $\mathscr{P}_2^{n-1}$. Consider some v of depth n. Since $\boldsymbol{\phi}$ is feasible for $\mathscr{P}_1^n$ and there is exactly one edge $e^+(v;X)$ in X whose origin is v, then

$$\begin{aligned}\boldsymbol{\phi}_{e^+(v;X)} &= q_v + \sum_{e\in E^-(v;H)} \boldsymbol{\phi}_e - \sum_{e\in E^+(v;H)\setminus X} \boldsymbol{\phi}_e \\ &= q_v + \sum_{e\in E^-(v;X)} \boldsymbol{\phi}_e + \sum_{e\in E^-(v;H)\setminus X} \boldsymbol{\phi}_e - \sum_{e\in E^+(v;H)\setminus X} \boldsymbol{\phi}_e \\ &= R(v;H,X) + \sum_{e\in E^-(v;X)} \boldsymbol{\phi}_e.\end{aligned}$$

Each node w in $E^-(v,X)$ must have pseudo-rank strictly less than v, so by induction:

$$\sum_{e\in E^-(v;X)} \boldsymbol{\phi}_e = \sum_{e\in E^-(v,X)} \sum_{u\in V(H):u\prec^X e} R(u;H,X).$$

We claim that the collection of sets

$$\mathcal{A} := \{V(L(e;X)) : e \in E^-(v;X)\}$$

is a partition of $V(L(v;X)) - v$. It is straightforward to verify that the union of the sets of $\mathcal{A}$ is equal to $V(L(v;X)) - v$. Suppose there were edges $e_1 = (w_1, W_1)$ and $e_2 = (w_2, W_2)$ in $E^-(v;X)$ such that $V(L(e_1;X))$ and $V(L(e_2;x))$ have a common node y. By Lemma Lemma SM3.1, $U(y;X)$ is a compath. Since H has disjoint reachability, every node of $U(y;X)$ must have at most one incoming edge. However, e_1 and e_2 are distinct incoming edges of v, both of which are in $U(y;X)$. This is a contradiction, so sets of $\mathcal{A}$ must be disjoint. Using this fact, we can rewrite the summation

$$\sum_{e\in E^-(v;X)} \boldsymbol{\phi}_e = \sum_{u:u\prec^X v} R(u;H,X).$$

Thus,

$$\begin{aligned}\boldsymbol{\phi}_{e^+(v,X)} &= R(v;H,X) + \sum_{u:u\prec^X v} R(u;H,X) \\ &= \sum_{u:u\preceq^X v} R(v;H,X).\end{aligned}$$

Thus, $\boldsymbol{\phi}$ satisfies the conservation constraints of $\mathscr{P}_2^n$. As in the base case, the capacity and non-negativity constraints are the same in $\mathscr{P}_1^n$ as in $\mathscr{P}_2^n$. Thus, $\mathscr{P}_1^n \subseteq \mathscr{P}_2^n$. This sequence of steps can be reversed to show that $\mathscr{P}_2^n \subseteq \mathscr{P}_1^n$. ∎

SM8. Proof of Theorem 6.3.

Statement of Theorem 6.3. Let there be an ADF-graph H, a set of edges $\mathscr{C}$ in $E(H)$, and a spanning in-forest X of H such that X is disjoint from $\mathscr{C}$. Let $\mathbf{q}$ and $\mathbf{a}$ be integral vectors of supplies and capacities respectively. If the non-degeneracy assumptions (stated at the beginning of section 6) hold then the convex hulls of integral solutions to $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$, $\mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$, and $\mathscr{P}_3(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$ all have dimension $|E \setminus X|$.

Proof of Theorem 6.3. First, note that there is a natural linear bijection between $\pi : \mathscr{P}_1 \to \mathscr{P}_3$ that preserves integrality. For a feasible solution $\boldsymbol{\lambda}$ to $\mathscr{P}_1$, $\pi(\boldsymbol{\lambda})$ is formed by omitting the variables f_e for each $e \in X$. In the opposite direction, consider a feasible solution $\boldsymbol{\lambda}$ to $\mathscr{P}_1$. The inverse $\pi^{-1}(\boldsymbol{\lambda})$ takes the same values as $\boldsymbol{\lambda}$ on the variables f_e for each $e \in E \setminus X$, and for $e \in X$ the value of the variables f_e are given by:

$$f_{e^+(v,X)} = \sum_{w: w \preceq^X v} \left(q_w + \sum_{e \in E^-(w;H) \setminus X} f_e - \sum_{e \in E^+(w;H) \setminus X} f_e \right) \qquad \forall v \in V(H)$$

Theorem 6.1 guarantees that this preserves feasible solutions, and it is easy to see that this mapping preserves integrality. This implies that the dimension of the convex hull of integral solutions to $\mathscr{P}_1$ and $\mathscr{P}_2$ is the same that of $\mathscr{P}_3$.

Since $\mathscr{P}_3$ has $|E \setminus X|$ variables, the dimension of these polyhedra is at most $|E \setminus X|$. We can identify $|E \setminus X| + 1$ integral affinely independent feasible solutions in $\mathscr{P}_1$ (or $\mathscr{P}_2$). This set of solutions contains $|E \setminus X|$ solutions that correspond to the edges of $E \setminus X$, as well as one additional solution. We first define a solution $\boldsymbol{O}$ in which the flow assigned the edge $e = (v, W)$ is given by:

$$f_e(\boldsymbol{O}) := \begin{cases} \sum_{w: w \preceq^X v} q_w & \text{if } e \text{ is in } X, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that this solution satisfies the constraints of $\mathscr{P}_2$, which implies that it is a feasible solution of $\mathscr{P}_1$.

We now construct a solution $\boldsymbol{\lambda}^e$ corresponding to each edge e in $E \setminus X$. Order the edges of $E \setminus X$ in a total ordering $\prec^*$ that is consistent with the natural partial ordering $\prec^H$. We will then construct the solutions corresponding to each arc one at a time, following the ordering $\prec^*$ and starting with the minimal arc. The solution $\boldsymbol{\lambda}^e$ for each arc e is constructed in such a way that e is $f_e(\boldsymbol{\lambda}^e) > 0$ and such that $e' \prec^* e$ for any edge e' of $F^+(\boldsymbol{\lambda}^e) \setminus X$. This implies that the solutions are affinely independent and that the projections of these solutions onto $\mathscr{P}_3$ are affinely independent. This fact is also useful in the construction of solutions in subsequent iterations. The details of the construction of $\boldsymbol{\lambda}^e$ for some edge $e = (v, W)$ are as follows. Lemma SM3.1 and Lemma SM3.2 imply that

$$P_e := e + \sum_{w \in W} U(w; X)$$

is a maximal compath. There are two cases:

1. There is some node $\rho \preceq^X v$ such that $q_\rho > 0$.
2. $q_\rho = 0$ for all $\rho \preceq^X v$.

Consider case 1. We define the corresponding solution $\boldsymbol{\lambda}^e = \mathbf{O} + \boldsymbol{\ell}(P_e) - \boldsymbol{\ell}(U(v; X))$. Note that $\mathbf{O}$ has no capacitated edges with positive flow, so it is trivially true that $E(P_e)$ shares no capacitated edges in $F^+(\mathbf{O})$. Note also that because there is some node $\rho \preceq^X v$ such that $q_\rho > 0$, then

$$\sum_{w: w \preceq^X v} q_v > 0.$$

From the construction of $\mathbf{O}$, we can then see that every edge in the compath $U(v; X)$ is assigned a positive flow in $\mathbf{O}$. Thus, by Lemma SM3.4, $\boldsymbol{\lambda}^e$ is a feasible solution.

Note that e is the only edge of $E \setminus X$ that is assigned a positive flow, so it is trivially true that all of the edges $E \setminus X$ that receive positive flow are less than or equal to e in the natural ordering on H.

For case 2, let ω be a minimal node of $L(v; X)$. From the non-degeneracy assumptions we know there is some node y such that $y \preceq^H \omega$ and $q_y > 0$, and we know that $q_\omega = 0$ because $q_\rho = 0$ for all $\rho \prec^X e$. Thus, there must be an edge η into the node ω. Since ω is a minimal node in the ordering $\prec^X$, η is an edge of $E \setminus X$. The solution $\boldsymbol{\lambda}^\eta$ was constructed in a previous iteration of this process because $\eta \prec^H e$. By construction, edge η is in $F^+(\boldsymbol{\lambda}^\eta)$, while $f_\xi(\boldsymbol{\lambda}^\eta) = 0$ for any edge ξ of $E \setminus X$ such that $\xi \succ^H \eta$. Thus, the edge $e^+(\omega, X)$ must receive positive flow in $\boldsymbol{\lambda}^\eta$ because there is a positive amount of flow on an incoming edge of ω, and there is zero flow on all other outgoing edges of ω. Thus, by Lemma SM3.3 there is a maximal compath $P_{e^+(\omega,X)}$ formed from edges of $F^+(\boldsymbol{\lambda}^\eta)$ starting at the edge $e^+(\omega, X)$. Then, by Lemma SM3.1 and Lemma SM3.4, the solution

$$\boldsymbol{\theta} := \boldsymbol{\lambda}^\eta + \boldsymbol{\ell}\left(U(\omega; X)\right) - \boldsymbol{\ell}\left(P_{e^+(\omega,X)}\right)$$

is a feasible solution in which all the edges of $U(\omega; X)$ are assigned a positive quantity of flow.

Note that it is still true in $\boldsymbol{\theta}$ that any edge of $F^+(\boldsymbol{\theta}) \setminus X$ is less than or equal to η in the natural ordering on H. Further note that in the solution $\boldsymbol{\theta}$, positive flow is assigned to all edges of $U(v; X)$ because $U(v; X)$ is a subgraph of $U(\omega; X)$. Thus, again applying Lemma SM3.1 and Lemma SM3.4, the solution

$$\boldsymbol{\lambda}^e := \boldsymbol{\lambda}^\eta + \boldsymbol{\ell}\left(U(\omega; X)\right) - \boldsymbol{\ell}\left(P_{e^+(\omega,X)}\right) + \boldsymbol{\ell}\left(P_e\right) - \boldsymbol{\ell}\left(U(v; X)\right)$$

is feasible, and has positive flow assigned to the edge e. Since $\eta \prec^H e$ and e is the only edge of $E \setminus X$ that is given more flow in $\boldsymbol{\lambda}^e$ than $\boldsymbol{\theta}$, it is true that $e' \prec^H e$ for any edge e' in $F^+(\boldsymbol{\lambda}^e) \setminus X$. This completes the construction. ∎

SM9. Additional Results for Subsection 6.2. This section provides three theorems that describe the conditions under which the constraints of the F-graph flow problem are facet-defining and the conditions under which these constraints are redundant. Together, these three theorems imply Theorem 6.4. Throughout this section, the constraints (3.1),(3.2), and (3.3) of $\mathscr{P}_1$ will be referred to as conservation, capacity and non-negativity constraints respectively; (6.1),(6.2), and (6.3) of $\mathscr{P}_2$ and (6.4),(6.5), and (6.6) of $\mathscr{P}_3$ will likewise be referred to by these names. We make use of the notation defined in section SM2.

THEOREM SM9.1. *Let there be an ADF-graph H, a vector of supply quantities $\boldsymbol{q}$, a set of capacitated edges $\mathscr{C}$, a vector of demand quantities $\boldsymbol{a}$, and a spanning in-forest X of $E(H)$ that contains no capacitated edges. Let there be some edge $\eta = (v, W)$ in $E(H) \setminus X$. If there is some $\rho \in W$ such that:*

1. *$q_\rho = 0$;*
2. *η is the only incoming edge of ρ;*

then the non-negativity constraint for the edge η in the polytope $\mathscr{P}_1(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a})$ is dominated by the conservation constraint of the node ρ and the non-negativity constraints on the outgoing edges of ρ. Otherwise, this constraint is facet-defining for the convex hull of integral solutions.

This statement also holds if $\mathscr{P}_1(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a})$ is replaced by $\mathscr{P}_2(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a}, X)$ or $\mathscr{P}_3(H, \boldsymbol{q}, \mathscr{C}, \boldsymbol{a}, X)$.

Proof. In the forward direction, let there be some (possibly infeasible or fractional) solution $\boldsymbol{\phi}$ that satisfies the conservation constraint for ρ and the non-negativity constraints for the edges in $E^+(\rho, H)$ as defined for the polytope $\mathscr{P}_1$. Then,

$$\sum_{e \in E^-(\rho;H)} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(\rho;H)} f_e(\boldsymbol{\phi}) = q_\rho.$$

Since η is the only incoming edge of ρ, this can be rewritten as

$$f_\eta(\boldsymbol{\phi}) = q_\rho + \sum_{e \in E^+(\rho;H)} f_e(\boldsymbol{\phi}).$$

Since $\boldsymbol{\phi}$ satisfies the non-negativity constraints on the edge $E^+(\rho; H)$ this implies that $f_\eta(\boldsymbol{\phi}) \geq 0$.

Similarly, let there be some (possibly infeasible or fractional) solution $\boldsymbol{\phi}$ that satisfies the conservation constraint for ρ and the non-negativity constraints for the edges in $E^+(\rho, H)$ as defined for the polytope $\mathscr{P}_2$. Then,the conservation constraint for ρ states that

$$f_{e^+(\rho;W)}(\boldsymbol{\phi}) = \sum_{w : w \preceq^X v} \left(q_w + \sum_{e \in E^-(w;H) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(w;H) \setminus X} f_e(\boldsymbol{\phi}) \right).$$

Since η is the only incoming edge of ρ and η is not in X, ρ has no preceding nodes in X. Therefore, this can be rewritten as

$$f_{e^+(\rho;W)}(\boldsymbol{\phi}) = f_\eta(\boldsymbol{\phi}) - \sum_{e \in E^+(\rho;H) \setminus X} f_e(\boldsymbol{\phi}),$$
$$\sum_{e \in E^+(\rho;H)} f_e(\boldsymbol{\phi}) = f_\eta(\boldsymbol{\phi}).$$

Again, the non-negativity constraints on the outgoing edges of ν imply that $f_\eta \geq 0$.

The argument in the forward direction for $\mathscr{P}_3$ is nearly identical. Let there be some (possibly infeasible or fractional) solution $\boldsymbol{\phi}$ that satisfies the conservation constraint for ρ and the non-negativity constraints for the edges in $E^+(\rho, H) \setminus X$ as defined for the polytope $\mathscr{P}_3$. Then,the conservation constraint for ρ states that

$$0 \leq \sum_{w : w \preceq^X \rho} \left(q_w + \sum_{e \in E^-(w;H) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(w;H) \setminus X} f_e(\boldsymbol{\phi}) \right).$$

As before, η is the only incoming edge of ρ and η is not in X, so ρ has no preceding nodes in X and we can rewrite this as:

$$0 \leq f_\eta(\boldsymbol{\phi}) - \sum_{e \in E^+(\rho;H) \setminus X} f_e(\boldsymbol{\phi}),$$
$$\sum_{e \in E^+(\rho;H) \setminus X} f_e(\boldsymbol{\phi}) \leq f_\eta(\boldsymbol{\phi}),$$

and the non-negativity constraints on the outgoing edges of ν imply that $f_\eta \geq 0$.

In the opposite direction, let η be an edge of $E(H) \setminus X$ and suppose that there is no node ρ such that η is the only incoming arc. We can construct a set of affinely-independent solutions of size $|E(H) \setminus X|$ in which $f_\eta = 0$. In fact, we claim that with very slight modifications, the procedure given in the proof of Theorem 3 can provide these solutions. Start with $\mathbf{O}$, as defined in the previous proof. Order the arcs of $E \setminus (X + \eta)$ in an ordering that is consistent with the natural partial ordering on H, and consider each arc $\xi = (v', W')$ starting with the minimal arc. The same two cases apply. If there exists some node $y \preceq^X \eta$ such that $q_y \geq 0$, then construct the solution $\boldsymbol{\lambda}^\xi$ as before. If such a node does not exist, then we construct the solution almost the same as before, but more care is taken as to which incoming edge we select. Again, let ω be a minimal node of $L(v'; X)$. As before, ω must have an incoming edge ζ in $E(H) \setminus X$. If $\zeta \neq \eta$, then we construct the solution $\boldsymbol{\lambda}^\eta$ by modifying the solution $\boldsymbol{\lambda}^\zeta$ as before. If $\zeta = \eta$, then by assumption ω must have another incoming edge κ in $E(H) \setminus X$, and we construct the solution $\boldsymbol{\lambda}^\eta$ by modifying the solution $\boldsymbol{\lambda}^\kappa$ as before.

By the same arguments as before, the projection of these solutions on to $\mathscr{P}_3$ is also affinely independent. Further note that in the construction of the solution $\boldsymbol{\lambda}^\xi$, positive flow is only assigned to edges that were considered in previous iterations of this procedure. Since no iteration is run for the edge η, none of these solutions assign positive flow on the edge η. ∎

In some cases, the capacity constraint on an edge can be dominated by the constraints corresponding the preceding subgraph. If sufficient flow cannot be routed to the origin node of an edge, then it could be that no feasible solution meets the capacity constraint. In this case, those preceding constraints dominate the capacity constraint. On the other hand, the capacity constraint is facet-defining whenever it is possible to route enough flow to the origin node.

THEOREM SM9.2. *Let there be an ADF-graph H, a vector of supply quantities $\boldsymbol{q}$, a set of capacitated edges $\mathscr{C}$, a vector of demand quantities $\boldsymbol{a}$, and a spanning in-forest X of $E(H)$ that contains no capacitated edges. Let η be an edge in $\mathscr{C}$. Let f^* be the value to the optimization problem* $\max\{f_\eta(\boldsymbol{\phi}) : \boldsymbol{\phi} \in \mathscr{P}_1(L(\eta; H), \boldsymbol{q}, \mathscr{C} - \eta, \boldsymbol{a})\}$. *If $f^* \leq a_\eta$, then the subset of constraints that correspond to the subgraph polytope $\mathscr{P}_1(L(\eta; H), \boldsymbol{q}, \mathscr{C} - \eta, \boldsymbol{a})$ dominate the capacity constraint for η. Otherwise, the constraint is facet-defining for the convex hull of integral solutions. This theorem is also true if $\mathscr{P}_1$ is replaced by $\mathscr{P}_2$ or $\mathscr{P}_3$.*

Proof. The forward direction is straightforward.

In the backwards direction, suppose that $f^* > a_\eta$. Then we can construct $|E(H) \setminus X|$ integral solutions that satisfy this constraint at equality. Let $\mathbf{a} + \boldsymbol{\ell}(\eta)$ be a vector $\mathbf{a}'$ of capacities where $a'_e = a_e$ for all $e \neq \eta$ and $a'_\eta = a_\eta + 1$. Lemma 4.3 and Lemma 5.4 imply that there exists an integral feasible solution $\boldsymbol{\phi}$ to $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a} + \boldsymbol{\ell}(\eta))$ such that:

- $f_\eta(\boldsymbol{\phi}) = a_\eta + 1$,
- $f_e(\boldsymbol{\phi}) = 0$ for all e in $E \setminus X$ where $e \npreceq^H \eta$.

We use a similar strategy as in previous proofs, and construct feasible solutions one at a time in an iterative procedure. In this proof, we divide this procedure into two stages. In each iteration the first stage, we define a solution $\boldsymbol{\lambda}^\xi$ corresponding to some edge ξ of $F^+(\boldsymbol{\phi}) \setminus (X + \eta)$. In each iteration of the second stage, we define a solution $\boldsymbol{\lambda}^\xi$ corresponding to an edge ξ of $F^0(\boldsymbol{\phi}) \setminus X$. We take care that across both stages, no solutions differ from $\boldsymbol{\phi}$ on the amount of flow assigned to some edge ξ until the iteration corresponding to this edge is reached. This guarantees that these solutions are affinely independent (and that their projections onto $\mathscr{P}_3$ are also affinely independent).

Order the edges of $F^+(\boldsymbol{\phi}) \setminus X$ in an ordering consistent with $\prec^H$. Consider each edge $\xi = (v, W)$ one at a time, starting with the maximum edge. Let P_ξ be a compath in the edges $F^+(\boldsymbol{\phi})$ whose first edge is ξ (such a compath must exist by Lemma SM3.3). Define a solution

$$\boldsymbol{\theta} := \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi) + \boldsymbol{\ell}(U(v;X)).$$

By Lemma SM3.4, this solution is a feasible solution of $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a} + \boldsymbol{\ell}(\eta))$. There are two cases: either η is in P_ξ or η is not in P_ξ. If η is in P_ξ then $f_\eta(\boldsymbol{\theta}) = a_\eta$, so that $\boldsymbol{\theta}$ is a feasible solution of $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$. In this case, we can let $\boldsymbol{\lambda}^\xi = \boldsymbol{\theta}$. If η is not in P_ξ then $f_\eta(\boldsymbol{\theta}) = a_\eta + 1$. Then let P_η be a compath in $F^+(\boldsymbol{\theta})$ starting at η (Lemma SM3.3 again ensures existence), let ρ be the origin node of η and let

$$\begin{aligned}\boldsymbol{\lambda}^\xi &:= \boldsymbol{\theta} - \boldsymbol{\ell}(P_\eta) + \boldsymbol{\ell}(U(\rho;X)) \\ &= \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi) + \boldsymbol{\ell}(U(v;X)) - \boldsymbol{\ell}(P_\eta) + \boldsymbol{\ell}(U(\rho;X)),\end{aligned}$$

which is a feasible solution for $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a} + \boldsymbol{\ell}(\eta))$ by Lemma SM3.4. Since η is not an edge of X, then it must be true that $f_\eta(\boldsymbol{\lambda}^\xi) = f_\eta(\boldsymbol{\phi}) - 1$, so that $\boldsymbol{\lambda}^\xi$ is a feasible solution for $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$.

In either case, note that $f_\xi(\boldsymbol{\lambda}^\xi) = f_\xi(\boldsymbol{\phi}) - 1$. Suppose that $f_\xi(\boldsymbol{\lambda}^\psi) \neq f_\xi(\boldsymbol{\phi})$ for some edge ψ in $E(H) \setminus X$ where $\psi \neq \xi$ and $\psi \neq \eta$. Then, ξ must be in the compath P_ψ or in the compath P_η. The compath P_η starts at η and consists of edges from $F^+(\boldsymbol{\phi})$. By definition of $\boldsymbol{\phi}$, $f_e(\boldsymbol{\phi}) = 0$ for all e in $E \setminus X$ where $e \not\preceq^H \eta$. This implies that η is the only edge of P_η that is not in X. Since $\xi \neq \eta$, it must be true that ξ is an edge in P_ψ. This implies that $\psi \prec^H \xi$, which in turn implies that $\boldsymbol{\lambda}^\psi$ was constructed in a later iteration than $\boldsymbol{\lambda}^\xi$. In other words, at the point in time when the solution $\boldsymbol{\lambda}^\xi$ is constructed, it is the only solution that differs from $\boldsymbol{\phi}$ in the amount of flow assigned on the edge ξ. Note also that $f_e(\boldsymbol{\phi}) = f_e(\boldsymbol{\lambda}^\xi)$ for any edge e in $F^0(\boldsymbol{\phi}) \setminus X$.

Next, consider the arcs of $F^0(\boldsymbol{\phi})$. Order these arcs in an ordering that agrees with the natural ordering $\prec^H$. Consider each arc ξ, starting with the minimum arc. We claim that there is a maximal compath P_ξ such that:

1. The edge ξ is an edge of P_ξ;
2. For any edge $e \in P_\xi$ such that $e \not\preceq^H \xi$, the edge e is in X;
3. The set of edges $E(P_\xi)$ is disjoint from $F^+(\boldsymbol{\phi}) \cap \mathscr{C}$;
4. The compath P_ξ starts at a node that has an outgoing edge ψ in $F^+(\boldsymbol{\phi})$.

If the origin node of ξ has a positive quantity of resource, then it is straightforward to verify that the path $U(e;X)$ is a compath that meets these conditions. Suppose that this origin node has no resource present. By the non-degeneracy conditions (given at the beginning of section 6), there must exist some path P_0 from some node ρ to the origin node of ξ where $q_\rho > 0$. Let $e^* = (v, W)$ be the minimal edge of P_0 such that $E(U(e^*; P_0 + \xi))$ is disjoint from $F^+(\boldsymbol{\phi})$. Consider the case that $v = \rho$, i.e. e^* is the outgoing edge of ρ in P_0. Since $q_\rho > 0$, then there must be another outgoing edge ψ in $F^+(\boldsymbol{\phi})$. Consider the case that e^* is not the outgoing edge of ρ in P_0. Then there exists some edge e' that precedes e^* in the path P_0; this edge must be in $F^+(\boldsymbol{\phi})$ from the definition of e^*. Since v has an incoming edge with positive flow in $\boldsymbol{\phi}$, it must also have an outgoing edge ψ in $F^+(\boldsymbol{\phi})$. Thus, in either case, $U(e^*; P_0)$ is a path in $F^0(\boldsymbol{\phi})$ whose origin node has an outgoing edge ψ in $F^+(\boldsymbol{\phi})$. By Lemma SM3.5, there exists a maximal compath P_ξ starting at v such that $E(U(e^*, P_0 + \xi)) \subseteq E(P_\xi) \subseteq E(U(e^*; P_0 + \xi)) \cup X$. Since ξ is the maximal edge of $P_0 + \xi$, it is an edge of $U(e^*, P_0 + \xi)$, so requirement 1 is satisfied by P_ξ. All of the edges in P_0 are less than or equal to ξ and are not in $F^+(\boldsymbol{\phi}) \cap \mathscr{C}$, so P_ξ also satisfies requirement 2 and 3. By definition, the compath starts at a node that has an outgoing edge ξ in $F^+(\boldsymbol{\phi})$, so requirement 4 is also satisfied. Let P_ψ be a compath starting at

ψ in $F^+(\boldsymbol{\phi})$ (the existence of this compath is guaranteed by Lemma SM3.1). Then, we can define a solution

$$\boldsymbol{\theta} := \boldsymbol{\phi} + \boldsymbol{\ell}(P_\xi) - \boldsymbol{\ell}(P_\psi),$$

which is a feasible solution for $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a} + \boldsymbol{\ell}(\eta))$ by Lemma SM3.4. As before, if η is an edge of P_ψ, then $\boldsymbol{\theta}$ is a feasible solution for $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$ and we let $\boldsymbol{\lambda}^\xi = \boldsymbol{\theta}$. If η is not in $E(P_\psi)$,

$$\begin{aligned}\boldsymbol{\lambda}^\xi &:= \boldsymbol{\theta} - \boldsymbol{\ell}(P_\eta) + \boldsymbol{\ell}(U(v, X))\\ &= \boldsymbol{\phi} + \boldsymbol{\ell}(P_\xi) - \boldsymbol{\ell}(P_\psi) - \boldsymbol{\ell}(P_\eta) + \boldsymbol{\ell}(U(v; X)).\end{aligned}$$

which is a feasible solution for $\mathscr{P}_1(H, \mathbf{q}, \mathscr{C}, \mathbf{a})$ for the same reasons as in the first stage.

Similar to before, note that $f_\xi(\boldsymbol{\lambda}^\xi) = f_\xi(\boldsymbol{\phi}) + 1$. We also claim that $\boldsymbol{\lambda}^\xi$ is the first solution constructed in this process that differs from $\boldsymbol{\phi}$ in the amount of flow assigned to ξ. Let $\boldsymbol{\lambda}^\xi$ be the solution corresponding to some ξ constructed in this second stage. Let $\boldsymbol{\lambda}^\psi$ be some other solution corresponding to an edge ψ such that $f_\xi(\boldsymbol{\lambda}^\psi) \neq f_\xi(\boldsymbol{\phi})$. Since ξ is assigned zero flow in the solution $\boldsymbol{\phi}$, then ξ must be assigned more flow in $\boldsymbol{\lambda}^\psi$ than in $\boldsymbol{\phi}$. In any solution constructed in the first stage, only edges of X receive more flow than in $\boldsymbol{\phi}$. Thus, $\boldsymbol{\lambda}^\psi$ is a second-stage solution. In a second-stage solution, $\boldsymbol{\lambda}^\psi$ only assigns more flow than $\boldsymbol{\phi}$ to edges of X and edges of a compath P_ψ. Thus, ξ is in the compath P_ψ. This compath is defined so that every edge of P_ψ is an edge of X or is less than or equal to ψ. Thus, ξ must be less than ψ. This would imply that the solution $\boldsymbol{\lambda}^\xi$ was constructed in a previous iteration than $\boldsymbol{\lambda}^\psi$. ◻

Theorem SM9.1 and Theorem SM9.2 describe the conditions under which the non-negativity and capacity constraints for edges of $E(H) \setminus X$ are facet-defining. In $\mathscr{P}_1$ and $\mathscr{P}_2$, the remaining inequality constraints are the non-negativity constraints corresponding to the edges of X. For $\mathscr{P}_3$, the conservation constraints are the constraints that have not yet been handled. There is a correspondence between these constraints. The non-negativity constraint of the edge $e^+(v; X)$ for some node v holds at equality in some solution $\boldsymbol{\phi}$ if and only if the conservation constraint at v holds at equality in the projection of $\boldsymbol{\phi}$ on to the polytope $\mathscr{P}_3$. For this reason, the conservation constraints in $\mathscr{P}_3$ are facet-defining when the non-negativity constraints on edges of X are facet-defining for $\mathscr{P}_1$ and $\mathscr{P}_2$. These conditions are described in Theorem SM9.3. We use the notation that $\mathbf{q} + \boldsymbol{\ell}(v)$ is the vector of quantities $\mathbf{q}'$ such that

$$q'_w = \begin{cases} q_w & \text{if } w \neq v,\\ q_w + 1 & \text{if } w = v.\end{cases}$$

THEOREM SM9.3. *Let there be an ADF-graph H, a vector of supply quantities $\boldsymbol{q}$, a set of capacitated edge $\mathscr{C}$, a vector of demand quantities $\boldsymbol{a}$, and a spanning in-forest X of $E(H)$ that contains no capacitated edges. Let v be any node in $V(H)$, and let*

$$f^* = \max \left\{ \sum_{w : w \preceq^X v} \left(\sum_{e \in E^+(w) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^-(w) \setminus X} f_e(\boldsymbol{\phi}) \right) : \right.$$
$$\left. \boldsymbol{\phi} \in \mathscr{P}_1(L(v; H) + E^+(v; H), \boldsymbol{q} + \boldsymbol{\ell}(v), \mathscr{C}, \boldsymbol{a}) \right\}.$$

Note that the value of f^* will remain unchanged if $\mathscr{P}_1$ is replaced with $\mathscr{P}_2$ or $\mathscr{P}_3$. Consider the following cases:

1. *There exists a destination node ρ of $\mathrm{e}^+(v,X)$ such that $q_\rho = 0$ and $\mathrm{e}^+(v,X)$ is the only incoming edge of ρ.*
2. *The value f^* is less than or equal to $\sum_{w:w\preceq^X v} q_w$.*
3. *Neither case 1 nor case 2 holds.*

If case 1 holds then the non-negativity constraint of $\mathrm{e}^+(v,X)$ is dominated by the non-negativity constraints of the outgoing edges of ρ and the conservation constraint on ρ. If case 2 holds then the non-negativity constraint of $\mathrm{e}^+(v,X)$ is dominated by the constraints of $\mathscr{P}_1(L(v;H)+E^+(v;H),\boldsymbol{q},\mathscr{C},\boldsymbol{a})$ excluding the non-negativity constraint on $\mathrm{e}^+(v,X)$. If case 3 holds, then the non-negativity constraint on $\mathrm{e}^+(v,X)$ is facet-defining for the convex hull of integral solutions. This statement remains true if $\mathscr{P}_1$ is replaced by $\mathscr{P}_2$.

A similar statement holds true for $\mathscr{P}_3$: if case 1 holds then the conservation constraint at v is dominated by the conservation constraint on the node ρ and the non-negativity constraints of the outgoing edges of ρ. If case 2 holds then the conservation constraint at v is dominated by the constraints of $\mathscr{P}_3(L(v;H)+E^+(v;H),\boldsymbol{q},\mathscr{C},\boldsymbol{a})$ excluding the conservation constraint at v. If case 3 holds, then the conservation constraint at v is facet-defining for the convex hull of integral solutions.

Proof. **Case 1.** For $\mathscr{P}_1$ and $\mathscr{P}_2$, the proof of this statement when case 1 holds is nearly identical to the proof of the forward direction of Theorem SM9.1, so we will only provide a proof for the statement for $\mathscr{P}_3$. Let there be some edge $e^+(v;X)$ in X that is the only incoming edge of a node ρ. Let $\boldsymbol{\phi}$ be a (possibly infeasible or fractional) solution to $\mathscr{P}_3(H,\mathbf{q},\mathscr{C},\mathbf{a},X)$ that satisfies the conservation constraint at ρ and the non-negativity constraints on the outgoing edges of ρ. Then

$$0 \le \sum_{w:w\preceq^X \rho} \left(q_w + \sum_{e\in E^-(w;H)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w;H)\setminus X} f_e(\boldsymbol{\phi}) \right).$$

Since v is the node immediately preceding ρ in X, the right-hand-side can be rewritten:

$$\begin{aligned} 0 \le & \sum_{w:w\preceq^X v} \left(q_w + \sum_{e\in E^-(w;H)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w;H)\setminus X} f_e(\boldsymbol{\phi}) \right) \\ & + \left(q_\rho + \sum_{e\in E^-(\rho;H)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(\rho;H)\setminus X} f_e(\boldsymbol{\phi}) \right). \end{aligned}$$

By the assumptions in this case, $q_\rho = 0$ and the set $E^-(\rho;H)\setminus X$ is empty. Then

$$0 \le \sum_{w:w\preceq^X v} \left(q_w + \sum_{e\in E^-(w;H)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w;H)\setminus X} f_e(\boldsymbol{\phi}) \right) - \sum_{e\in E^+(\rho;H)\setminus X} f_e(\boldsymbol{\phi}).$$

Applying the non-negativity constraints on the outgoing edges of ρ,

$$0 \le \sum_{w:w\preceq^X v} \left(q_w + \sum_{e\in E^-(w;H)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w;H)\setminus X} f_e(\boldsymbol{\phi}) \right).$$

Thus, the conservation constraint on the node v is dominated by the conservation constraint of ρ and the non-negativity constraints on the edges of ρ. This completes the proof of the statement when condition 1 holds.

Case 2. The proof of the statement for $\mathscr{P}_2$ when case 2 holds is as follows. First, we claim that non-negativity constraint for the edge $e^+(v; H)$ of the polytope $\mathscr{P}_2\left(L(v;h) + E^+(v;H), \mathbf{q} + \ell(v), \mathscr{C}, \mathbf{a}\right)$ is a redundant constraint of this polytope. Let $\boldsymbol{\phi}$ be a feasible (possibly fractional) solution to this polytope. Then, by definition of f^* and the assumptions of the case,

$$\begin{aligned}
\sum_{w:w\preceq^X v} q_w &\geq \sum_{w:w\preceq^X v}\left(\sum_{e\in E^+(w)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^-(w)\setminus X} f_e(\boldsymbol{\phi})\right)\\
0 &\leq \sum_{w:w\preceq^X v}\left(q_w + \sum_{e\in E^-(w)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w)\setminus X} f_e(\boldsymbol{\phi})\right)\\
1 &\leq \sum_{w:w\preceq^X v}\left(q'_w + \sum_{e\in E^-(w)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w)\setminus X} f_e(\boldsymbol{\phi})\right)\\
1 &\leq f_{e^+(v;X)}(\boldsymbol{\phi}).
\end{aligned}$$

Thus, there is no solution $\boldsymbol{\phi}$ in which the non-negativity constraint on the edge $e^+(v;H)$ holds at equality, and this constraint is redundant.

Let there be some (infeasible, possibly fractional) solution $\boldsymbol{\theta}$ in which all constraints of the polytope $\mathscr{P}_2\left(L(v;h) + E^+(v;H), \mathbf{q}, \mathscr{C}, \mathbf{a}\right)$ other than the non-negativity constraint on $e^+(v,;X)$ are satisfied. We claim that the solution

$$\boldsymbol{\phi} := \boldsymbol{\theta} + \boldsymbol{\ell}(e^+(v;X))$$

is a feasible solution to $\mathscr{P}_2\left(L(v;h) + E^+(v;H), \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a}\right)$. Note that the only constraint of this polytope in which $e^+(v;X)$ appears the non-negativity constraint on this edge is the conservation constraint on v. Now,

$$\begin{aligned}
f_{e^+(v;X)}(\boldsymbol{\phi}) &= f_{e^+(v;X)}(\boldsymbol{\theta}) + 1\\
&= \sum_{w:w\preceq^X v}\left(q_w + \sum_{e\in E^-(w)\setminus X} f_e(\boldsymbol{\theta}) - \sum_{e\in E^+(w)\setminus X} f_e(\boldsymbol{\theta})\right) + 1\\
&= \sum_{w:w\preceq^X v}\left(q'_w + \sum_{e\in E^-(w)\setminus X} f_e(\boldsymbol{\phi}) - \sum_{e\in E^+(w)\setminus X} f_e(\boldsymbol{\phi})\right).
\end{aligned}$$

So $\boldsymbol{\phi}$ satisfies the conservation constraint on v. As we already showed, the non-negativity constraint on $e^+(v;X)$ is redundant. The solution $\boldsymbol{\phi}$ satisfies all other constraints, so it will satisfy this non-negativity constraint as well. Thus, $\boldsymbol{\phi}$ solution is a feasible solution for $\mathscr{P}_2\left(L(v;h) + E^+(v;H), \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a}\right)$ excluding the non-negativity constraint on $e^+(v;X)$. From the previous arguments,

$$f_{e^+(v;X)}(\boldsymbol{\phi}) \geq 1.$$

This implies that

$$f_{e^+(v;X)}(\boldsymbol{\theta}) \geq 0.$$

Thus, any feasible solution that satisfies the constraints of $\mathscr{P}_3(L(v;H)+E^+(v;H), \mathbf{q}, \mathscr{C}, \mathbf{a})$ excluding the non-negativity constraint on $e^+(v;H)$ will also satisfy the non-negativity constraint.

The proof of the statement for $\mathscr{P}_3$ when case 2 holds is similar. Since

$$f^* \leq \sum_{w:w \preceq^X v} q_w$$

then for any feasible solution $\boldsymbol{\phi}$ of $\mathscr{P}_3(L(v;H) + E^+(v;H), \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a}, X)$,

$$\begin{aligned}
\sum_{w:w \preceq^X v} q_w \geq \sum_{w:w \preceq^X v} & \left(\sum_{e \in E^+(w) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^-(w) \setminus X} f_e(\boldsymbol{\phi}) \right), \\
0 \leq \sum_{w:w \preceq^X v} & \left(q_w + \sum_{e \in E^-(w) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(w) \setminus X} f_e(\boldsymbol{\phi}) \right), \\
1 \leq \sum_{w:w \preceq^X v} & \left(q'_w + \sum_{e \in E^-(w) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(w) \setminus X} f_e(\boldsymbol{\phi}) \right),
\end{aligned}$$

for any feasible solution $\boldsymbol{\phi}$ in the polytope. Thus, the conservation constraint on v is never met at equality in this polytope, so the constraint can be removed. Since v is a maximal node in this polytope, the parameter q'_v does not appear in any other constraints. Thus, the resulting polytope is equal to $\mathscr{P}_3(L(v;H) + E^+(v;H), \mathbf{q}, \mathscr{C}, \mathbf{a})$ with the conservation constraint on v removed. Thus, for any solution of $\boldsymbol{\phi}$ of $\mathscr{P}_3(L(v;H) + E^+(v;H), \mathbf{q}, \mathscr{C}, \mathbf{a})$, it is true that

$$\sum_{w:w \preceq^X v} q_w \geq \sum_{w:w \preceq^X v} \left(\sum_{e \in E^+(w) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^-(w) \setminus X} f_e(\boldsymbol{\phi}) \right).$$

This proves that the constraints of $\mathscr{P}_3(L(v;H) + E^+(v;H), \mathbf{q}, \mathscr{C}, \mathbf{a})$ excluding the conservation constraint on v dominate this conservation constraint.

Case 3. Suppose that case 3 holds. It is implied by Lemma 4.3 and Lemma 5.4 that there exists an integral solution $\boldsymbol{\phi}$ for $\mathscr{P}_2(H, \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a})$, such that $f_e(\boldsymbol{\phi}) = 0$ for any e in $E \setminus (X \cup E^+(v;H))$ where $e \not\prec^H v$. Due to the assumptions of this case, there exists a solution that additionally satisfies:

$$\begin{aligned}
\sum_{w:w \preceq^X v} \left(\sum_{e \in E^+(w;H) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^-(w;H) \setminus X} f_e(\boldsymbol{\phi}) \right) &= \sum_{w:w \preceq^X v} q'_w, \\
\sum_{w:w \preceq^X v} \left(q'_w + \sum_{e \in E^-(w;H) \setminus X} f_e(\boldsymbol{\phi}) - \sum_{e \in E^+(w;H) \setminus X} f_e(\boldsymbol{\phi}) \right) &= 0, \\
f_{e^+(v,X)}(\boldsymbol{\phi}) &= 0.
\end{aligned}$$

In a similar fashion to previous constructions, we use this solution to construct $|E(H) \setminus X|$ integral feasible solutions $\mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$ in which the non-negativity constraint on $f_{e^+(v;X)}$ is satisfied to equality. These are produced in a three-stage process. In each iteration of the first stage, a solution corresponding to an edge of

$(F^+(\boldsymbol{\phi}) \cup E^+(v; H)) \setminus X$ is constructed. In the second stage, solutions corresponding to an edge of $F^+(\boldsymbol{\phi}) \setminus (X \cup E^+(v; H))$ are constructed. In the final stage, a solution corresponding to an edge of $F^0(\boldsymbol{\phi}) \setminus X$ is constructed. As in the proof of Theorem SM9.2, the solutions are constructed such that $\boldsymbol{\lambda}^e$ is the first solution in this process which assigns an amount of flow on e that differs from that in $\boldsymbol{\phi}$. This ensures that the solutions are affinely independent and that the projections of these solutions into $\mathscr{P}_3$ are affinely independent as well. For any solution $\boldsymbol{\phi}$ for $\mathscr{P}_2$, the flow $f_e^+(v; X)(\boldsymbol{\phi})$ is equal to zero if and only the conservation constraint holds at equality in the projection of $\boldsymbol{\phi}$ onto $\mathscr{P}_3$, so this provides the proof for the statements for all three polytopes.

For each edge in $(F^+(\boldsymbol{\phi}) \cup E^+(v; H)) \setminus X$, let P_ξ be a maximal compath in $F^+(\boldsymbol{\phi})$ starting at ξ (such a compath must exist by Lemma SM3.3) and let

$$\boldsymbol{\lambda}^\xi = \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi).$$

We claim that this is a feasible solution for $\mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$. The non-negativity constraints and edge capacity constraints are clearly preserved. The conservation constraint for any node $\rho \neq v$ is preserved, since the maximal compath $P_{e^+(v;X)}$ has either one incoming and outgoing edge from ν or no incoming edges and no outgoing edges. The conservation constraint at v is satisfied because this solution has one less unit of outgoing flow at v that $\boldsymbol{\theta}$ and the quantity of supply at the node is decreased by one. It is clear that $f_{e^+(v;X)}(\boldsymbol{\lambda}^\xi) = 0$. By definition of $\boldsymbol{\phi}$, any edge assigned positive flow that is greater than an outgoing edge of v must either be an edge in X. Thus, we can see that solution $\boldsymbol{\lambda}^\xi$ only differs from ϕ in the flow assigned to ξ and the flow assigned to edges of X.

Order the edges of $F^+(\boldsymbol{\phi}) \setminus (E^+(v; H) \cup X)$ in some total ordering consistent with the natural partial ordering on H. Consider each edge ξ in order, starting with the maximum edge. Let P_ξ be a maximal compath in $F^+(\boldsymbol{\phi})$ starting at ξ (such a compath must exist by Lemma SM3.3). Let y be the origin node of ξ. Define a solution

$$\boldsymbol{\theta} = \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi) + \boldsymbol{\ell}(U(y; X)).$$

By Lemma SM3.1 and Lemma SM3.4, $\boldsymbol{\theta}$ is a feasible solution of $\mathscr{P}_2(H, \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a})$. Furthermore, it is clear that

$$\begin{aligned} f_{e^+(v;X)}(\boldsymbol{\theta}) &\leq f_{e^+(v;X)}(\boldsymbol{\phi}) + 1 \\ &\leq 1. \end{aligned}$$

There are two cases: either $f_{e^+(v;X)}(\boldsymbol{\theta}) = 1$ or $f_{e^+(v;X)}(\boldsymbol{\theta}) = 0$. Consider the former case. Then, let $P_e^+(v; X)$ be a complete compath in $F^+(\boldsymbol{\theta})$ starting with the edge $e^+(v; X)$ (such a compath must exist due to Lemma SM3.3). By a similar justification as that given in the first stage construction,

$$\begin{aligned} \boldsymbol{\lambda}^\xi &:= \boldsymbol{\theta} - \boldsymbol{\ell}(P_{e^+(v;X)}), \\ &= \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi) + \boldsymbol{\ell}(U(y; X)) - \boldsymbol{\ell}(P_{e^+(v;X)}). \end{aligned}$$

is a feasible solution for $\mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$. Note also that $\boldsymbol{\lambda}^\xi(e^+(v; X)) = 0$, so the non-negativity constraint of $e^+(v; X)$ holds to equality.

Consider the case that $f_{e^+(v,X)}(\boldsymbol{\theta}) = 0$. Then it must be true that $f_\eta(\boldsymbol{\phi}) \geq 1$ for some outgoing edge η of v not in X, because

$$\begin{aligned}\sum_{e \in E^+(v) \setminus X} f_e(\boldsymbol{\theta}) &= 1 + q_v + \sum_{e \in E^-(v)} f_e(\boldsymbol{\theta}) \\ &\geq 1.\end{aligned}$$

Let P_η be a complete compath starting at η in $F^+(\boldsymbol{\theta}_\xi)$ (such a compath must exist due to Lemma SM3.3). Then, let

$$\begin{aligned}\boldsymbol{\lambda}^\xi &:= \boldsymbol{\theta} - \boldsymbol{\ell}(P_\eta) \\ &= \boldsymbol{\phi} - \boldsymbol{\ell}(P_\xi) + \boldsymbol{\ell}(U(y;X)) - \boldsymbol{\ell}(P_\eta).\end{aligned}$$

For similar reasons as the previous case, $\boldsymbol{\lambda}_\xi$ is a feasible solution for $\mathscr{P}_2(H, \mathbf{q}, \mathscr{C}, \mathbf{a}, X)$ and it is clear that $f_{e^+(v,X)}(\boldsymbol{\theta}_\xi) = 0$.

We can show that in either case, each second-stage solution $\boldsymbol{\lambda}^\xi$ is the first solution that differs from $\boldsymbol{\phi}$ in the amount of flow assigned on ξ. Suppose that some solution $\boldsymbol{\lambda}^\psi$ constructed in another iteration corresponding to an edge $\psi = (v', W')$ differs from $\boldsymbol{\phi}$ in the amount of flow on ξ. As previously discussed, each first-stage solution differs from $\boldsymbol{\phi}$ only on the flow assigned to the outgoing edge of v and to edges of X. Thus, $\boldsymbol{\lambda}^\psi$ must be a second- or third-stage solution. If $\boldsymbol{\lambda}^\psi$ is a third-stage solution, then ψ is constructed after ξ. If $\boldsymbol{\lambda}^\psi$ is a second-stage solution, it is constructed by making three modifications to $\boldsymbol{\phi}$: flow on compath P_ψ is decreased, flow on the compath $U(v';X)$ is increased, and flow is decreased on some path $P_{e'}$ starting on an outgoing edge of v. The compath P_ψ only contains edges of $F^+(\boldsymbol{\phi})$ that are greater than or equal to ψ. The compath $U(v';X)$ only contains edges of X. Finally, for similar reasons as in the first stage, every edge of $E(P_{e'})$ other than e' is an edge of X. If e' is in X, then all edges of $P_{e'}$ are edges of X, while if e' is not in X then it is one of the edges whose corresponding solution was constructed in the first stage. Thus, $\boldsymbol{\lambda}^\psi$ differs from $\boldsymbol{\phi}$ only in flow assigned to edges that are either greater than ψ or are in X. Thus, ξ must be greater than ψ, which implies that the solution $\boldsymbol{\lambda}^\psi$ would be constructed after the solution $\boldsymbol{\lambda}^\xi$.

Finally, order the edges of $F^0(\phi) \setminus X$ according to some total ordering that is consistent with the natural partial ordering on H. Take each edge ξ of this ordering, starting with the minimum edge. For the same reasons as in the previous theorem, we claim that there is a compath P_ξ such that:

1. The compath P_ξ contains the edge ξ;
2. For any arc $e \in P_\xi$ such that $e \not\succeq \xi$, then $e \in X$;
3. The edges $E(P_\xi)$ are in $F^0(\boldsymbol{\phi}) \cup X$;
4. The compath P_ξ starts at an node that has an outgoing edge ψ in $F^+(\boldsymbol{\phi})$.

Let P_ψ be a compath starting at ψ in $F^+(\boldsymbol{\phi})$ (such a compath must exist by Lemma SM3.3). ∎ Then, we can define a solution

$$\boldsymbol{\theta} := \boldsymbol{\phi} + \boldsymbol{\ell}(P_\xi) - \boldsymbol{\ell}(P_\psi).$$

This solution is a feasible solution for $\mathscr{P}_2(H, \mathbf{q} + \boldsymbol{\ell}(v), \mathscr{C}, \mathbf{a})$ by Lemma SM3.4. Similarly to the construction of the second-stage solutions, we can see that

$$\begin{aligned}f_{e^+(v,X)}(\boldsymbol{\theta}) &\leq f_{e^+(v,X)}(\boldsymbol{\phi}) + 1 \\ &\leq 1,\end{aligned}$$

and just as before, if $f_{e^+(v,X)}(\boldsymbol{\theta}_\xi) = 1$ then we define:

$$\boldsymbol{\lambda}^\xi = \boldsymbol{\theta} - \boldsymbol{\ell}(P_{e^+(v,X)}),$$

while if $f_{e^+(v,X)}(\boldsymbol{\theta}_\xi) = 0$, then there exists some outgoing edge η of v such that $\boldsymbol{\theta}_\xi(\eta) \geq 1$, and we let

$$\begin{aligned}\boldsymbol{\lambda}^\xi &= \boldsymbol{\theta} - \boldsymbol{\ell}(P_\eta)\\ &= \boldsymbol{\phi} + \boldsymbol{\ell}(P_\xi) - \boldsymbol{\ell}(P_\psi) - \boldsymbol{\ell}(P_\eta).\end{aligned}$$

for some compath P_η in $F^+(\boldsymbol{\theta})$ starting with the edge η. Again we can show that in either case, each third-stage solution $\boldsymbol{\lambda}^\xi$ is the first solution that differs from $\boldsymbol{\phi}$ in the amount of flow assigned on ξ. Suppose the solution to some other solution $\boldsymbol{\lambda}^\psi$ corresponding to an edge $\psi = (v', W')$ differs from $\boldsymbol{\phi}$ in the amount of flow on ξ. Since $\boldsymbol{\phi}$ assigns zero flow to the edge ξ, then $\boldsymbol{\lambda}$ must assign higher flow on this edge than $\boldsymbol{\phi}$ does. Again, each first-stage solution differs from $\boldsymbol{\phi}$ only on the flow assigned to the outgoing edge of v and to edges of X. Each second-stage solution only assigns higher flow than $\boldsymbol{\phi}$ on edges of X, while $\boldsymbol{\lambda}^\xi$ assigns higher flow on ξ than $\boldsymbol{\phi}$ does. Thus, $\boldsymbol{\lambda}^\psi$ must be a third-stage solution. The solution $\boldsymbol{\lambda}^\psi$ only assigns higher flow than $\boldsymbol{\phi}$ along the path P_ψ starting at ψ. By definition, every edge of P_ψ is either less than ψ or is in X. Thus, ψ is greater than ξ, which implies that $\lambda\psi$ is constructed in a later iteration than $\boldsymbol{\lambda}^\xi$. This completes the proof. ∎